RETHINKING KNOWLEDGE FOR TEACHING MATHEMATICS WITH TECHNOLOGY: TENSIONS, EXTENSIONS AND FUTURE AGENDA

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The integration of digital technology to promote mathematical learning remains challenging for teachers. The continuous difficulty in fully achieving technology's potential highlights the need to further examine and develop models addressing teachers' knowledge in relation to technology. In this Research Forum, we analyse existing technology-related knowledge models, exploring similarities and differences. We also consider how these models have been used in research, highlighting tensions driven by emerging technologies, and resulting extensions of the models. Finally, we discuss possible integrations across frameworks and the potential to include dimensions beyond knowledge. This discussion aims to deepen understanding of current models and suggest directions and a research agenda for future development.

INTRODUCTION

The integration of digital technology holds potential for enhancing mathematical learning, yet realizing this potential remains challenging. Teachers play a central role, with their professional knowledge strongly influencing this integration (Faggiano et al., 2021).

The persistent difficulty in fully realizing the potential of technology in education points to the relevance of further developing the existing theoretical models. One possible direction for such development lies in reconciling knowledge-based models with other approaches, through the construction of a framework that allows for joint implementation. This perspective is aligned with the work developed by Haspekian (2023), who integrates, into an instrumental approach, a specific framework that models teachers' practices into dimensions not restricted to mathematical knowledge. This is also the case of PTK (*Pedagogical Technology Knowledge*, Thomas & Hong, 2013) already incorporating teachers' personal orientations, and of KTMT (*Knowledge for Teaching Mathematics with Technology*, Rocha, 2020a) explicitly considering both external and internal contexts, the latter encompassing a wide range of teacher-specific aspects. Within this line of research, Tabach (2021) has drawn attention to the notion of competence, considering it indispensable for a meaningful and comprehensive

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integration of technology in teaching. Emerging technologies have turned a long-standing need for the development of theoretical frameworks into a critical necessity. These technologies have reshaped the relevance of existing knowledge domains and foregrounded new ones, leading some authors to propose alternative models (e.g., I-TPACK). This calls for a critical reflection on the adequacy of current frameworks, a discussion central to the AI-TecTeachers project, which brings together most members of this Research Forum (RF).

Drawing on our ongoing discussions and collaborations in several conferences and projects, we argue that reflecting on the existing knowledge models – namely their development, articulation with other frameworks, and adaptation to emerging technologies – is crucial to deepen our understanding of technology integration.

This reflection is the focus of this RF on the topic of *knowledge for teaching mathematics with technology*, guided by the following question and sub-questions:

What do existing knowledge models for teaching mathematics with technology enable us to understand, and how can their future development be envisaged in light of ongoing theoretical, technological, and educational challenges?

(a) How should knowledge models evolve to respond to the demands posed by emerging technologies?

(b) To what extent should future models integrate domains beyond knowledge?

(c) What is the potential of integrating different theoretical frameworks to develop richer and more coherent perspectives on technology integration?

KNOWLEDGE MODELS

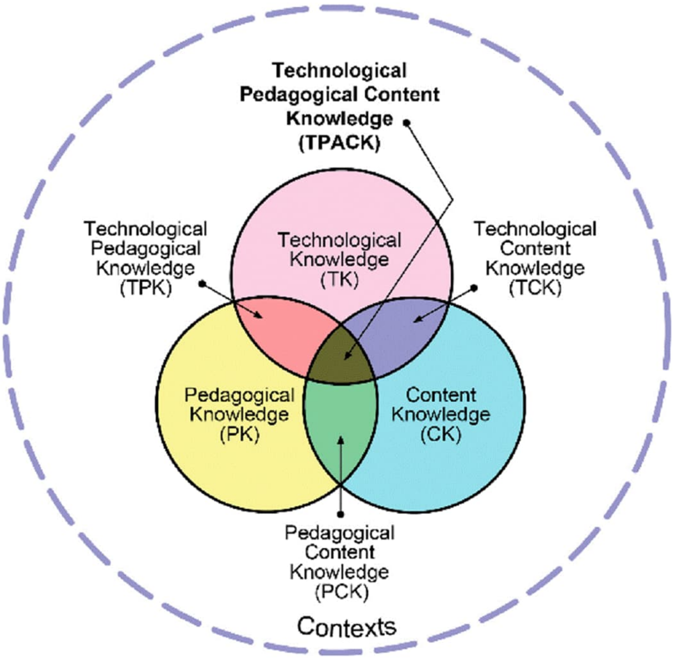
Shulman's PCK – Pedagogical Content Knowledge is a mark in the research on teachers' knowledge. Since then, several authors have developed refinements and new frameworks. The integration of technology raised new questions and led to the TPACK model (*Technological Pedagogical Content Knowledge*, Mishra & Koehler, 2006), inspired by PCK; the KTMT model (Rocha, 2020a), combining insights from research on professional knowledge with findings from studies on technology integration and emphasizing teachers' practice; and the PTK model (Thomas & Hong, 2013), incorporating instrumental genesis into a knowledge framework.

In this section we present three knowledge models with a specific focus on technology.

TPACK – Technological Pedagogical Content Knowledge

The TPACK framework emerged from Shulman's (1987) concept of PCK, which highlights the integration of content and pedagogy, i.e., understanding how specific subject matter can be organized, adapted, and taught effectively. Mishra and Koehler (2006) extended this idea by incorporating technology, recognizing that effective teaching requires not only knowledge of content and pedagogy but also knowledge about how to use technology to support and transform teaching and learning.

Figure 1: TPACK model by Mishra and Koehler (reproduced by permission of the publisher, © 2012 by tpack.org).



TPACK conceptualization (Figure 1) includes the three knowledge domains of *Content Knowledge* (CK), *Pedagogical Knowledge* (PK), and *Technological Knowledge* (TK), along with the intersections among them: *Pedagogical Content Knowledge* (PCK), *Technological Content Knowledge* (TCK), *Technological Pedagogical Knowledge* (TPK). CK refers to understanding the facts, concepts, theories, and structures of a discipline, while PK is generic and involves knowledge of teaching methods, learning processes, instructional goals, and strategies for students' assessment. TK encompasses the knowledge necessary to operate the technology. PCK is consistent with Shulman's conceptualization. TCK refers to understanding how technology and content influence each other. Teachers must not only be knowledgeable about the subject they teach but also understand how technology can transform the way that content can be presented and learned. TPK refers to understanding the features, and potential of different technologies in educational settings, as well as how teaching practices can be adapted or transformed through their use. TPACK is seen as "the connections, interactions, affordances, and constraints between and among content, pedagogy, and technology" (Mishra & Koehler, 2006, p. 1025).

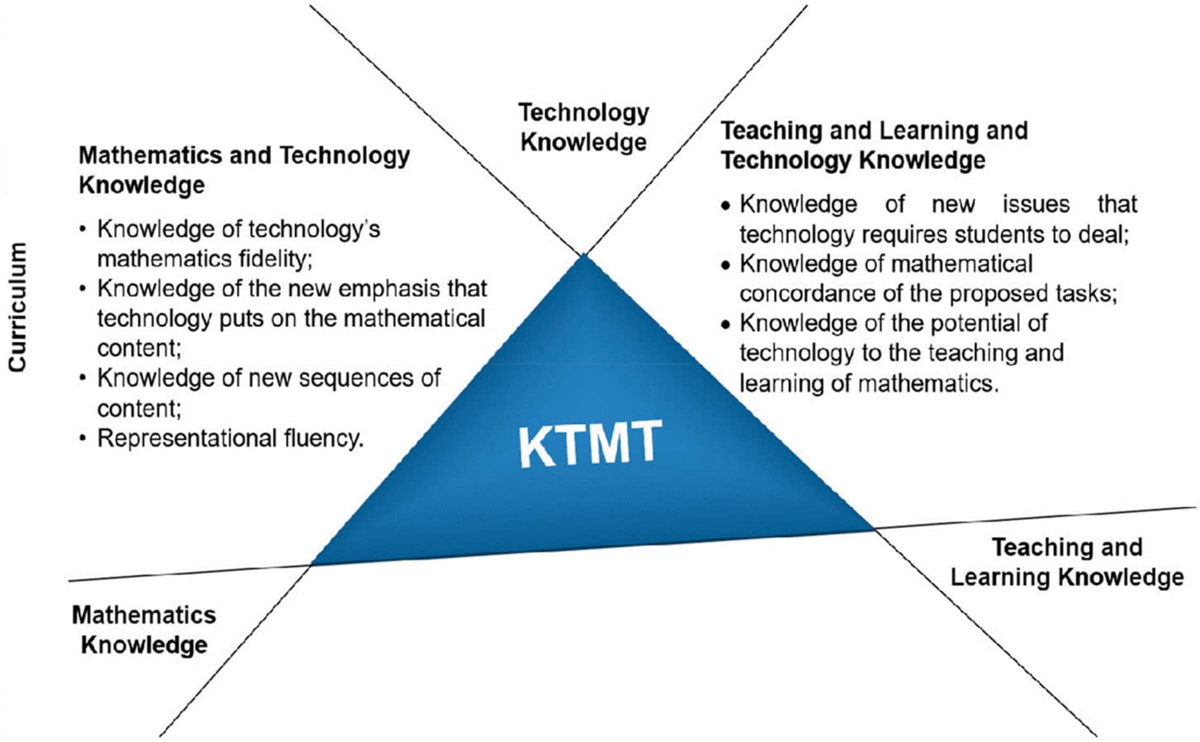
Context has evolved within the model. Initially absent, it was later acknowledged and, more recently, treated as a distinct knowledge domain, recognizing that contextual factors shape the practical application of TPACK in real classrooms (Mishra, 2019).

KTMT – Knowledge for Teaching Mathematics with Technology

The KTMT model by Rocha (2020a) builds on the work of Shulman (1987) and Mishra and Koehler (2006), while aiming to go beyond the TPACK framework. The model

(Figure 2) intends to integrate the existing research on teachers' knowledge and on technology integration into a single comprehensive model. With a particular focus on Mathematics teaching, the model includes base knowledge domains, two sets of inter-domains knowledge, and *Integrated Knowledge* (IK) in a pyramidal structure.

Figure 2: KTMT model by Rocha (2025).



The base knowledge domains are four (the last one being transversal and influential over the others): Mathematics, Teaching and Learning, Technology, and Curriculum and Context. Context includes the internal context, such as personal influences (e.g., teachers' beliefs), and external context, such as the school environment. The other domains, drawing from the earlier works of Shulman (1987), Ball et al. (2008), and Mishra and Koehler (2006), are understood basically in the same way. Inter-domain knowledge plays a central role in the model, distinguishing it from other knowledge frameworks. The KTMT model considers two sets of inter-domain knowledge: the Mathematics and Technology Knowledge (MTK), and the Teaching and Learning and Technology Knowledge (TLTK). MTK focuses on the impact of technology on mathematics, addressing how it enhances some elements while constraining others, and TLTK focuses on the impact of technology on the teaching and learning process, addressing how it enhances some approaches while constraining others. Both inter-domain knowledge areas are informed by research on technology integration, incorporating key findings of the research in the field. As so, MTK includes:

- knowledge of the technology's mathematical fidelity, meaning an understanding of how closely the output produced by technological tools aligns with the mathematical result (Dick, 2008);
- knowledge of the new emphases that technology places on mathematical content (e.g. knowing how the values represented on the coordinate axes can affect the appearance of the graph displayed by the technology) (Rocha, 2020b);
- knowledge of new sequences of content (Drijvers, 2013);
- knowledge of representations or *representational fluency*, including knowledge of how to navigate between various forms of representation and the choices that can be made to effectively illustrate or support mathematical concepts (Rocha, 2016; Zbiek et al., 2007).

TLTK includes:

- knowledge of new issues that technology requires students to deal with (e.g. what is the best strategy to look for a proper viewing window (Rocha, 2020b));
- knowledge of the mathematical concordance of the proposed tasks, i.e. the alignment between teachers' objectives and the mathematical concepts that students are actually engaging with in their learning (Zbiek et al., 2007);
- knowledge of the potential of technology to the teaching and learning of mathematics, i.e., knowledge of how to coordinate various types of tasks, leveraging the opportunities provided by technology, and allowing students to assume a more central role in their learning process (Dunham, 2000).

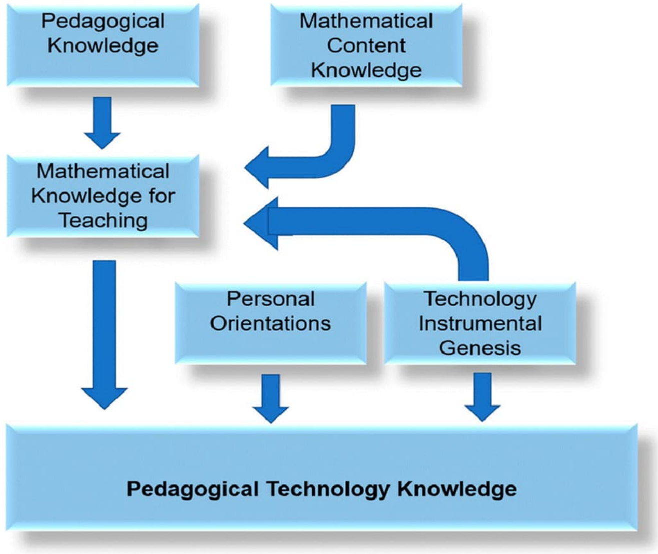
IK represents the final type of knowledge within the KTMT model, emerging from the interplay (and not intersection) among all the various knowledge domains.

PTK – Pedagogical Technology Knowledge

This model (Figure 3) adopts a very different perspective from the previous models, highlighting the use of technology and how the technological tool can be transformed into different instruments, in line with Rabardel (1995). However, it is possible to identify in this model, as in the previous ones, the influence of the work of Shulman and his notion of PCK.

Thomas and Hong (2013, p. 69) characterise the teachers' PTK as including "the principles, conventions and techniques required to teach mathematics through the technology". With its roots in the work of Shulman, this model considers as domains the *Mathematical Content Knowledge* and the *Pedagogical Knowledge*. However, instead of PCK, it considers the work of Ball et al. (2008) and their MKT – *Mathematical Knowledge for Teaching*. Besides these, the authors assume as important two domains that are not knowledge domains. That is the case of *Technology Instrumental Genesis,* assumed as highly relevant, and of *Personal Orientations* also with a strong impact recognised by the authors. All these domains contribute to PTK.

Figure 3: PTK model by Thomas and Hong (2013).



Differences and similarities across the knowledge models – discussing the models

The models presented here have several differences across them, but also several similarities. Some of these differences and similarities are related to the domains considered. We will get back to this in the section about the *Global Model*, here we discuss how the models conceived the development of the knowledge, addressing also how they are used in research.

The conceptualisation of the TPACK model is not so clear about how knowledge development is conceived. Some authors interpret the model in an integrative perspective and some others in a transformative one (Angeli & Valanides, 2009). As so, it is not clear if knowledge in two base domains (*Content*, *Pedagogy*, or *Technology*) results straightforward in knowledge in the intersection of these domains – the integrative perspective – or if it requires the development of some new knowledge – the transformative perspective. In the uses of the model in the literature, it is possible to find both interpretations.

KTMT has a pyramidal or a layer structure. As other models intending to conceptualise the knowledge needed to teach with technology, this model assumes a need for knowledge going beyond knowing how to operate the technology, valuing knowledge about the impact of technology in Mathematics and in the teaching and learning process (the two inter-domains knowledge). Instead of intersections as in TPACK, the inter-domain knowledge is a new knowledge developed from more than one *base knowledge* and integrating in its characterisation results from the research on technology integration. And the IK develops from the base and the inter-domains knowledge in a similar way. This is an on-going process with a continuous development of the knowledge in all the domains of the model. This means that although the model has a pyramidal or layer structure, the development on the three layers (the layer of the *base* *knowledge*, the layer of the *inter-domain knowledge*, and the layer of *integrated knowledge*) is not sequential. As so, teachers with *Integrated Knowledge* can still continue to develop their knowledge in one (or more) of the *inter-domains* or in one (or more) of the *base domains*. It also means that the development of knowledge in one domain does not result straightforwardly in the development of knowledge of another domain (namely at an upper level or layer). It is only by the development of knowledge resulting from the interaction of the new knowledge in one domain with knowledge in some other domain that new knowledge can result (e.g. some new knowledge about technology does not result immediately in new MTK, there is a need for some interaction, with the new knowledge on technology impacting the *Mathematics Knowledge* and resulting in the development of new knowledge – the new MTK).

Regarding PTK, the main difference is that the model incorporates instrumental genesis as central in technology integration. However, this model has not been operationalized in research to analyse teachers' knowledge in practice. The few existing studies seem to be based on quantitative approaches, using the model as a basis for the development of a questionnaire. In fact, as highlighted by Tabach (2021), this is a relatively uncommon approach in existing literature.

On the contrary, TPACK seems to be the most used knowledge model. TPACK is applied in both quantitative and qualitative studies. In the latter, TPACK is often employed holistically and articulated with other theoretical frameworks, functioning more as a general guiding idea than as an analytical model in which each knowledge domain is examined in depth and separately. In contrast, the KTMT framework, due to its strong connection to classroom practice, is more frequently used in qualitative approaches that place significant emphasis on the distinct domains it encompasses. Its structure allows for a more fine-grained analysis of teachers' practices and the multiple dimensions involved in technology use.

The continuous emergence of new frameworks – from PCK to TPACK, KTMT, and PTK – shows that teachers' knowledge is not a settled construct but a dynamic theoretical object that must be repeatedly rethought as mathematical practices, tools, and classroom activity evolve, making it a persistent central concern for both mathematics education research and the PME community.

Questions for discussion:

(1) What core assumptions about teachers' knowledge and teaching practice are shared by TPACK, KTMT, and PTK, and where do they diverge?

(2) How does each framework conceptualize the relationship between content, pedagogy, and technology?

(3) What conceptual tensions and limitations emerge from a comparative analysis of these frameworks, and how do they inform potential extensions or refinements?

DEVELOPMENT OF KNOWLEDGE MODELS

In this section we start by foregrounding how established teacher-knowledge models are used with contemporary technologies, and what *tensions* this use makes visible. The focus is to show how the models operate as lenses for design and analysis, and how emergent technologies pressure their assumptions. Then, we show how the emergence of Artificial Intelligence (AI) has led scholars to explore teachers' professional knowledge specific to the use of AI-based digital technology and suggest extensions to existing knowledge models.

Enactment of different models with traditional and emergent technologies

We begin with TPACK enactments around dynamic digital tools; then bring in two PTK enactments, one in the context of computational thinking/programming at scale, and one within a Community-of-Inquiry professional development design; and close with two recent KTMT enactments with Generative AI (GenAI).

In a modelling and simulation program (Quarder et al., 2025), TPACK is specialised into four tightly coupled knowledge dimensions (theory, task, process, intervention) and woven across preparation, practice and reflection. In this enactment, TPACK both specifies what knowledge teachers need when simulations organise the modelling process, and structures how the course must interweave that knowledge so teachers can diagnose process difficulties and choose interventions that sustain autonomy and mathematical verification. The key tension is that TK cannot be handled in isolated sessions; it must be distributed across design, enactment and reflection to orchestrate simulation-based activity with integrity.

A complementary topic/tool-specific TPACK enactment follows pre-service teachers who plan, implement, video-analyse and re-design lessons on polygons with GeoGebra (Kartal & Çınar, 2022). TPACK becomes a performance lens to distinguish demonstration from inquiry (e.g. whether students "touch" the tool, use dragging to reason about invariants, or use measurement as a springboard for argument). This use of the framework surfaces an enactment gap: robust TCK/TPK plans often revert to teacher-centred routines under classroom time and management pressures, indicating the need for explicit links to evidence of practice (e.g., video analysis) to support plan-to-action translation.

A national-scale enactment (Turgut et al., 2024) operationalises PTK at scale for computational thinking (CT) in mathematics by Rasch-validating three components: mathematical knowledge for teaching with technology (MKTwT), instrumental genesis (IG), and personal orientations (PO). Distinguishing personal from professional IG, and capturing PO (e.g. pragmatic vs. epistemic value, time costs, by-hand skills) explains why uptake can stall despite positive attitudes. Three *tensions* follow: (i) a persistent gap between positive orientations and the hard work of integration reframing TK as a dual IG to be developed personally and professionally; (ii) years of experience or generic professional development are weak predictors, while school level matters, signalling integration as situated, content-bound, and likely to need grade-specific CT articulations; (iii) large-scale measures show where work stalls but need to be paired with in-situ qualitative analyses of orchestration/design to explain why reported confidence coexists with limited peer support or code-level reasoning.

A second enactment situates PTK in a Community-of-Inquiry (CoI) professional development (Anabousy & Tabach, 2022): in-service teachers iterate plan - enact - video-reflect - redesign inquiry cycles. PTK components show strong mutual relations (notably, confidence and IG). PTK improves across components except content knowledge, with differential gain by seniority and prior technology integration level, indicating that social–organisational support is needed to consolidate PTK in practice.

We finally report on two KTMT enactments that analyse teachers' interaction with Generative AI. In a design–interaction–reflection trajectory (Bernardi et al., 2025), KTMT locate TK in the practical craft of prompting, MTK in teachers' fidelity judgments, and TLTK in the orchestration of step-wise guidance and collective critique, while Curriculum & Context domain regulates what can count as mathematically acceptable versus productively debatable. Used as an analytic scaffold, the model shows how knowledge becomes visible in enactment, and how GenAI reconfigures relations among TK, MTK and TLTK.

Another study using KTMT (Noster et al., 2024) analyses teachers' ChatGPT logs: prompting strategies are read as TK, the co-presence of correct and plausible but wrong outputs through MTK's fidelity lens; and follow-ups (eliciting, justifying, or repeating prompts) as fragile or emerging TLTK. KTMT provides a vocabulary for linking observable interaction patterns to knowledge domains and exposes task dependence and prompting choices as conditions for both fidelity and pedagogy.

The enactment cases considered offer a coherent view of what knowledge models reveal when technologies reshape mathematical activity and highlight a common set of *tensions*. TPACK, when specialised and phase-embedded, shows that TK must be woven through design, enactment and reflection and operationalised by genres (e.g., simulation-based modelling; dynamic geometry), and that teachers' orchestration of dynamic systems (not just pre-lesson plans) is pivotal. PTK, with its measurable MKTwT–IG–PO components, foregrounds instrumental genesis and beliefs, explaining why positive orientations can coexist with shallow uptake, why integration is situated rather than generic, and how communities of inquiry help convert confidence into enacted competence. KTMT, in GenAI settings, shifts attention to the fragility/negotiability of fidelity, the status of prompting as specific TK, and the need to conceptualise teaching as the orchestration of a conversational agent within mathematical norms. These tensions have direct *implications* for model development: making negotiated fidelity explicit; representing prompting strategies and verification routines within TK (and challenging what is only TK or includes other types of knowledge); theorising orchestration as managing dialogue with and around digital systems; adopting phase-embedded and performance-sensitive designs when activity is reorganised by technologies.

These *tensions* set up the next section where we examine *extensions* addressing these pressures, review TPACK variants that integrate reliability and ethics, and consider links with instrumental and documentational approaches to understand how teachers learn to orchestrate AI-mediated mathematics responsibly and effectively.

The case of TPACK adaptations to AI

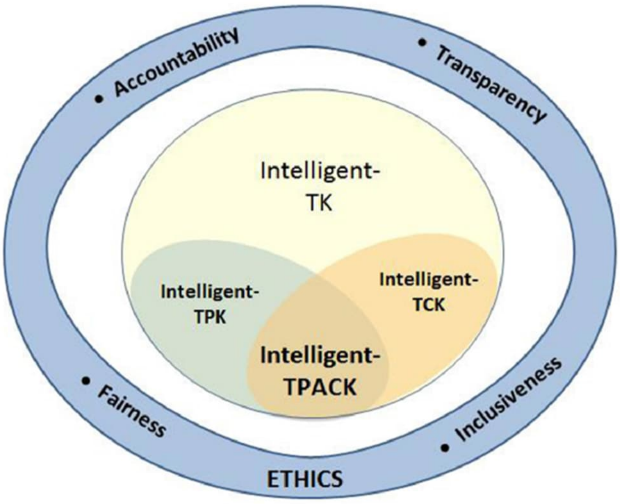
Based on the literature, we discuss how the *tensions* posed by AI-based emerging technologies are leading to *extensions* to the existing knowledge models, and outline the main characteristics of the approaches being adopted.

Most of the knowledge models developed to account for the specificities of AI-based technology are extensions of the TPACK model. In what follows, we focus on three extensions and reconceptualizations of TPACK developed to account for the unique characteristics of AI – its agency, content-generation capabilities, and inherent ethical complexities (Celik, 2023; Chiu, 2025; Lan et al., 2025).

All the authors argue that the nature of AI requires an evolution of the TPACK model. The core limitations identified across the studies include: (1) *tool agency and autonomy*: AI tools act as "partners" or "agents" that generate content, make decisions, and interact with users autonomously, shifting the teachers' role from operating a tool to managing an intelligent system; (2) *epistemic challenges*: since GenAI can produce sophisticated, human-like text and media (Chiu, 2025, p. 4191), teachers need to critically curate, validate, and contextualize AI-generated content, guarding against bias and "hallucinations"; (3) *ethical complexity*: privacy, fairness, transparency, and accountability are central to the responsible use of AI in the classroom; (4) *dynamic collaboration*: the interactive nature of AI opens possibilities for new forms of human-AI collaboration (Chiu, 2025, p. 4189), thus teachers need knowledge to strategically orchestrate human-AI synergies.

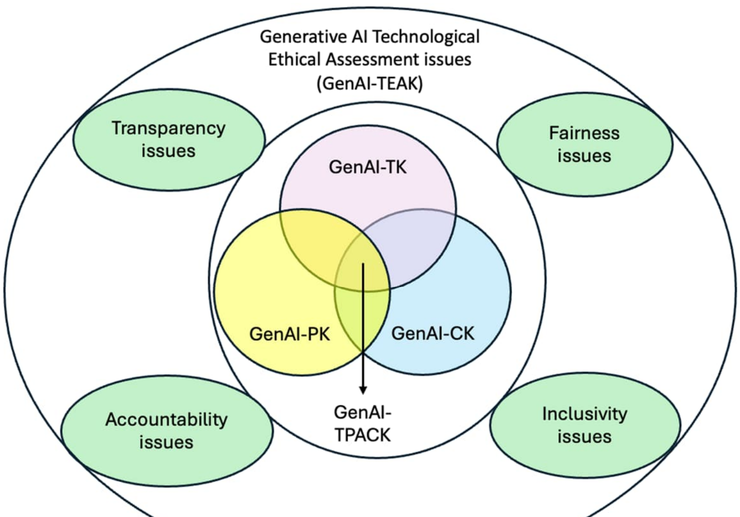
In response to these limitations, each article proposes a modified framework. While sharing a common goal, they differ in scope, structure, and methodological approach.

Figure 4: The Intelligent-TPACK framework proposed by Celik (2023).



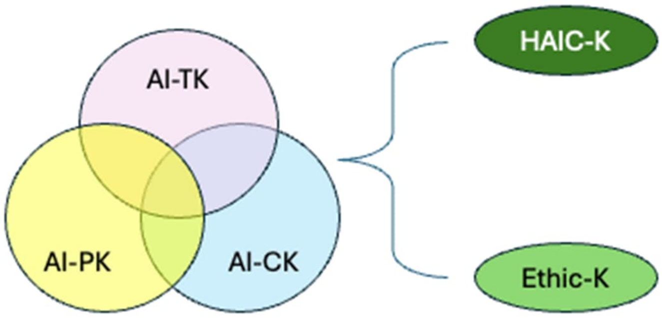
Celik's (2023) framework focuses on a broad range of AI-based tools, including chatbots, intelligent tutoring systems (ITS), AI-enhanced dashboards, and automated assessment systems. The proposed *Intelligent-TPACK framework* (Figure 4) extends the traditional model by adding a distinct fifth knowledge component: *ethics* that measures a teacher's ability to assess AI-based decisions concerning four ethical aspects: transparency, fairness, accountability, and inclusiveness.

Figure 5: The GenAI-TPACK framework, adapted from Lan et al. (2025).



Building on similar premises, Lan et al. (2025) narrow the focus to generative AI and propose the *GenAI-TPACK framework* (Figure 5), which incorporates *Generative AI Technological Ethical Assessment Knowledge* (GenAI-TEAK) as an essential component. This domain is defined as the ability to evaluate the ethical implications of GenAI tools regarding transparency, accountability, fairness, and inclusivity.

Figure 6: The I-TPACK framework, adapted from Chiu (2025).



Rather than an extension of the TPACK, Chiu (2025) proposes a radical reconceptualization, termed the *Intelligent-TPACK* (I-TPACK) framework (Figure 6) to bridge the gap between AI literacy and AI competency. The I-TPACK framework introduces five, interdependent knowledge domains: *AI-Technological Knowledge* (AI-TK) that includes prompt engineering, algorithmic awareness, and system auditing; *AI-Content Knowledge* (AI-CK) needed to critically curate, validate, and contextualize AI-generated disciplinary content, including bias detection; *AI-Pedagogical Knowledge* (AI-PK) necessary to design AI-augmented learning experiences, redesign assessments, and structure human-AI co-agency; *Human-AI Collaborative Knowledge* (HAIC-K), a novel domain focused on structuring interactions where teachers, students, and AI coordinate cognitive tasks; and *Ethical Knowledge* (EK), centralized as a core domain that permeates all other domains, addressing data privacy, algorithmic justice, and other moral dimensions. Chiu's framework is unique in its granularity, its redefinition of the original TPACK components for the AI context, and its introduction of HAIC-K. It positions teachers as "augmented pedagogy designers" who orchestrate complex human-AI synergies.

Development of knowledge models – discussion

The three studies with focus on AI provide a multi-layered understanding of the evolving teachers' knowledge base. They exhibit powerful commonalities in their foundational arguments while also offering unique contributions. Celik's (2023) and Lan et al.'s (2025) frameworks highlight the centrality of ethical knowledge as a core component. The two studies converge on acknowledging that TK serves as a necessary foundation that positively impacts the development of TPK, TCK, and ethical knowledge, which in turn predicts effective integration. Chiu's (2025) framework is the only one to formalize Human-AI Collaborative Knowledge (HAIC-K). This is a significant contribution, as it reframes AI from tool to partner, foregrounding teachers' professional knowledge for orchestrating human-AI collaboration, that is managing the dynamic interplay between human and AI in the classroom.

Questions for Discussion:

(1) How have emerging technologies prompted adaptations or extensions of TPACK, KTMT, and PTK in the context of mathematics teaching?

(2) In what ways do these adaptations highlight specific aspects of teachers' knowledge that are emphasized, transformed, or newly conceptualized within the extended or hybrid models?

(3) To what extent do the adaptations observed in knowledge models represent incremental refinements versus more fundamental conceptual shifts?

(4) How do technology-specific extensions (e.g., I-TPACK) balance the need for theoretical coherence with the risk of framework fragmentation?

COMBINING FRAMEWORKS – A POSSIBLE APPROACH?

While emerging technologies may highlight specific knowledge domains, frameworks themselves evolve continuously. Accordingly, we focus on how combined frameworks can offer more effective lenses to study the complexity of technology integration.

We address this point from two different perspectives: integrating different models in a single one and combining a knowledge model with other frameworks. Our focus is on the models and on discussing the potential of combining frameworks.

The Global Model – combining TPACK, KTMT, PTK

Rocha (2025) studied similarities and differences across TPACK, KTMT and PTK, knowledge models identified in a literature review (Rocha, 2023), and proposed the *Global Model*. Knowledge of Mathematics domain is present in all the models, although with different names: *Content Knowledge* for TPACK, a model that is not specific for Mathematics; *Mathematics Knowledge* for KTMT; and *Mathematical Content Knowledge* for PTK. Another domain present in all the models relates to pedagogical knowledge (*Pedagogical Knowledge* in the case of TPACK and PTK, and *Teaching and Learning Knowledge* in the case of KTMT). Although the names differ, in the case of the KTMT model it emphasizes teaching and learning dimensions, yet in all cases the underlying understanding is similar.

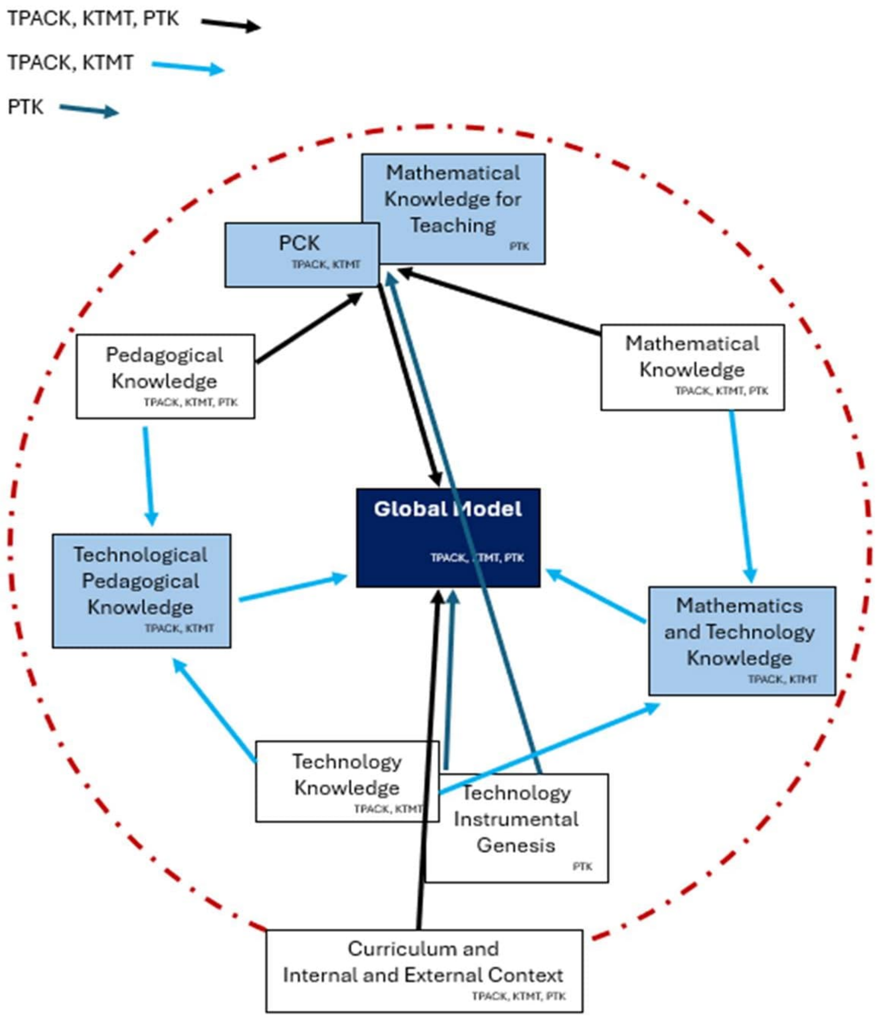
Some recognition of the relevance of contextual issues is also present in all the models, although in this case with significant differences.

TPACK includes *Contextual Knowledge*, referring to contextual circumstances, external to the teachers but with impact on their practices. KTMT considers *Curriculum and Context*. In this case, the context includes internal and external influences on the teachers, *Curriculum* being assumed as the prescribed curriculum and, as so, one of the external influences over the teachers. So, the understanding of context is more comprehensive in the case of KTMT, including the understanding assumed in TPACK that is basically related to external influences over the teachers, but going beyond it, namely in what relates to external influences, where it also includes the curriculum. The PTK model includes *Personal Orientations* without any reference to context. However, this domain is closely related to the understanding of internal context present in KTMT. Globally, we can assume that TPACK considers the external context of the teacher, PTK the internal context and KTMT both the internal and the external contexts.

A significant difference between the models is present in the domain related to technology. In the case of TPACK and KTMT it is considered a domain of *Technology Knowledge*. In the case of PTK it is considered *Technology Instrumental Genesis*. This means that TPACK and KTMT focus on a technical knowledge about how to operate the technology, but PTK, focusing on instrumental genesis, values the instrumentation and the instrumentalization, in line with Rabardel (1995). As so, it is not simply a matter of knowing what the technology does, but also of knowing what it does that can turn it into a useful instrument for Mathematics.

Globally, combining the three models and attending to the similarities and differences across them, results in the representation of the *Global Model* presented in Figure 7.

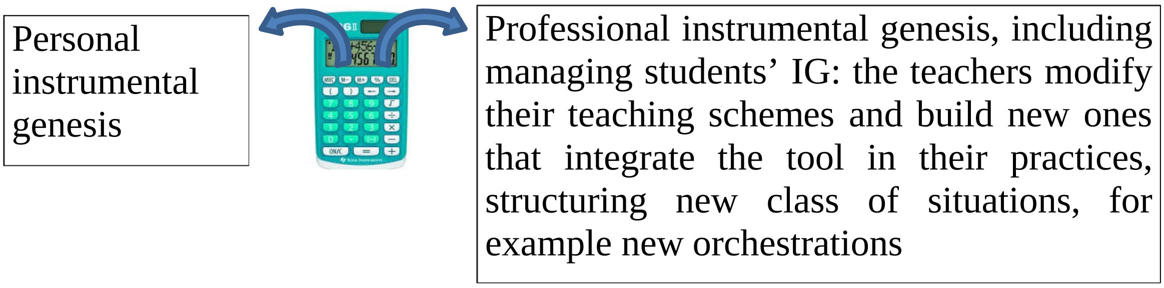
Figure 7: The *Global Model* © 2024 by Helena Rocha is licensed under CC BY 4.0. https://creativecommons.org/licenses/by/4.0/



TPACK and extensions of the Instrumental Approach to teachers

The *Instrumental Approach* (IA) used in mathematics education stems from a theoretical and cultural tradition distinct from TPACK foundations. It relies on Vérillon and Rabardel's (1995) work in cognitive ergonomics, thus, its roots trace back to Gérard Vergnaud (one of the founding figures of PME conferences). With the concept of *Instrumental Genesis* (IG), Rabardel (1995) highlighted the non-neutral impact of tools on conceptualizations, shedding light on phenomena observed in research on symbolic calculators (Artigue, 2002; Lagrange, 1999). With these tools, mathematics differs from traditional institutional content, this gap being later called *Instrumental Distance* (ID). To address both institutional and instrumental dimensions, the authors combined IA with the anthropological theory of didactics (Chevallard, 1992), giving rise to the *Instrumental approach in mathematics education* (IAME). Combined with other frameworks, IAME has led to theoretical conceptualizations extending beyond content knowledge alone. Initially centered on the students' instrumented learning, IAME progressively extended towards teachers, e.g., with the concept of *Instrumental* Orchestration (Trouche, 2004), the *Documentational Approach* (Gueudet & Trouche, 2009), or by applying the concept of IG directly to teachers (Haspekian, 2014), introducing the *Double IG* concept (Figure 8).

Figure 8: Double instrumental genesis of the teacher (Haspekian, 2014).



The integration of a new artefact raises questions falling under the professional IG such as which tasks, orchestrations, assessments to design, which common misconceptions and remedies to anticipate, or how long teaching concepts may take. These didactic reference points (*repères didactique*s) used in everyday teaching must be rebuilt. The bigger the ID, the greater the loss of such *teaching reference points* (RPs). Haspekian (2017) defined them as professional anchors that guide teacher's actions and are characterized by combining IAME with a teacher-centered framework, the Double Approach (DA) (Robert & Rogalski, 2002). This extension provides ground for drawing comparisons with the TPACK model.

In the DA framework, teaching practices are seen as a coherent multidimensional system that brings into play five components: cognitive, mediative, social, institutional, and personal. Haspekian situates the RPs in the personal component, and uses the other four to describe how they function as constituents that guide activity (Table 1). RPs thus go beyond content knowledge, they can be institutional and social (these components strongly determine teachers' practices); didactic (related to teachers' cognitive and mediative classroom choices); or biographical (explaining individual cognitive and mediative choices within the institutional and social constraints).

Table 1: Possible RP for analysing the personal component of teachers' practices.

Institutional and social dimensions	Institutional RP: constraints related to curricula, school hours, official recommendations…
	Social RP: constraints related to the school area and students' sociocultural backgrounds
Didactic dimensions	Cognitive RP: organization of contents to be taught and cognitive pathways planned for students.
	Mediative RP: organization in space, time, discourse accompanying sessions with students.
Biographic dimension (epistemology, representations, beliefs, culture…)	Epistemological RP: teachers' personal view of mathematics, its teaching, and learning, derived from personal representations/beliefs and the cultural context
	Pedagogical RP: teachers' personal view on teaching and learning in general, derived from personal representations/beliefs and the cultural context

ID and RPs impact the decision to integrate a new artefact, each dimension potentially weighing positively or negatively. A big ID may disrupt teachers' RPs but also be of cognitive interest (by supporting learning of particular mathematical concepts, offering visualizations, etc.). The perceived didactic legitimacy of the tool may encourage its integration despite the loss of RPs. The institutional dimension may hinder the integration if the ID is too big regarding the prescribed curriculum. Conversely, this dimension may favor integration if curricula legitimize the tool use.

How do the RPs framework relate to the TPACK model?

Both perspectives share the focus on teachers' use of technology to teach mathematics, and both have the potential to nurture professional development. Yet, the issues addressed are different. Unlike TPACK, RPs do not aim to categorize knowledge, but to analyze teachers' practices in order to understand the difficulties they meet as well as the strengths they rely on when they successfully integrate technology.

This difference leads to a more structural one. While TPACK seeks to categorize knowledge according to its nature, RPs rather seek to understand their functions, i.e., their effective instantiation in teaching practices: what role do teachers' beliefs and knowledge – conscious or unconscious, pedagogical, content-related, technological, and their intersections – play in teaching? Teachers may have PCK of "differentiated pedagogy" and strong convictions about its benefits for students, yet may not integrate it, missing RPs for its concrete implementation in classroom. Hence, the RPs lens seeks to address an explanatory, even predictive, approach to the dynamics at play, rather than a descriptive approach to a fixed and normative teacher knowledge model. Consequently, RPs are structured with categories that relate to their functions, while TPACK is structured by categories that relate to their nature.

Table 2: Identifying RPs within the TPACK, based on Tabach (2011) [left column].

Excerpts from Tabach (2011, p. 250): TPCK characteristics and operational description in teacher's utterances		Didactic RP according to the components of the Double Approach
TPCK characteristics (Mishra & Koehler 2006, p. 1029)	Type of evidence looked for – utterances which …	
An understanding of the representation of concepts using technology	Refer to algebraic concepts and their representation in a spreadsheet environment	Cognitive RPs and personal representations of the mathematical contents
Pedagogical techniques that use technologies in constructive ways to teach content	Include the teacher's pedagogical decisions which refer to the mathematics and the availability of Excel	Mediative and epistemological RPs of the contents
Knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face	Reflect the teacher's familiarity with ways students perceived concepts – correctly or incorrectly – in the presence of Excel	Cognitive RPs
Knowledge of students' prior knowledge and theories of epistemology	Reflect the teacher's familiarity with students' personal knowledge, difficulties and their theoretical sources	Cognitive and epistemological RPs of the contents
Knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones	Refer to actions taken by the teacher aimed at promoting mathematical knowledge with respect to technological affordances	Idem

The emphasis on instantiation is another key difference. RPs are not fixed pieces of knowledge but situational updates of knowledge, skills, and expertise, as well as more volatile elements such as judgments, perceptions, and intuitions specific to a teaching moment. For example, a math teacher says she cannot design assessments prior to sequences; she needs to take RPs on what students have actually engaged with during the sequence. While she certainly needs some PK or CK too, this illustrates the situational role of RPs, not given in advance nor fixed but specific to each class history.

Other researchers have also sought to move beyond the descriptive nature of TPACK to focus on more dynamic aspects. Some have done so by incorporating the concept of IG, as shown in the previous sections. Similarly, Tabach (2011) jointly used TPACK and instrumental orchestration to study the dynamic evolution of teaching practices, relating changes in teachers' expressed knowledge to changes in the observed orchestrations. In doing so, she highlighted several common points between the two frameworks. Drawing on her table of TPACK characteristics, we add the components of the DA to further the comparisons with RPs (Table 2). This shows that TPACK neither differentiate between choices and constraints, nor does it explicitly account for institutional and social dimensions that impact teachers' choices and activity. By contrast, it strongly emphasizes cognitive and mediating components, providing a holistic view of the different types of knowledge that may be needed in these components. These statements reflect that TPACK is not designed to explain teaching activity, it tackles the issue of technological integration in a different way from the RPs perspective.

Knowledge models developments by combining frameworks – discussion

Combining different knowledge models or combining a knowledge model with a different framework can be one option to develop more powerful analysis lens. However, the process is challenging and brings to the front several questions.

Questions for Discussion:

(1) What are the conceptual and practical benefits of combining knowledge frameworks for understanding technology integration in mathematics classrooms?

(2) To what extent can integrated frameworks address limitations of individual models, and where might tensions or inconsistencies emerge?

(3) What are the potential and limits of combining frameworks as a way forward?

BEYOND KNOWLEDGE

In this section we address another direction for future development of knowledge models: the integration of domains beyond knowledge, with a focus on teachers' competence.

Technological pedagogical content knowledge or knowledge for teaching mathematics with technology, specified in the discussed models and frameworks, is the basis or condition for developing learning processes. However, a teacher needs, beyond knowledge, the ability to incorporate or apply the knowledge to the teaching process and that in turn needs an orientation concerning the goals of the teaching and learning processes. Teaching and learning are (nearly) always two sides of one coin. These domains, knowledge, teaching abilities and orientation, can be encapsulated in the concept of teachers' competencies.

In this subsection we address a direction for future development which is beyond knowledge. We ask for competence models and focus on teachers' digital competencies. Next, we refer to different ways in which teachers may adopt technology or digital media to their classroom practices. This will set the ground to delve into digital competencies for teaching. Finally, we bring two conceptualisations within mathematics education for digital competencies for teaching mathematics.

The SAMR-Model

Teaching competencies concerning digital media can also be characterised by the changes in content that teachers aim to achieve using digital media compared to technology-free teaching. According to Puentedura's (2006) SAMR model, four types of teaching can be distinguished in this regard. The four letters SAMR stand for:

S - *Substitution*: Analogue media are replaced by digital media, whereby the content requirements for learners remain the same and no extended functions of digital media are used.

A - *Augmentation*: Analogue media are replaced by digital media, whereby the content requirements for learners remain the same, but the expanded functionalities of digital media are utilised.

M - *Modification*: Traditional tasks for learners are modified in such a way that new content-related requirements emerge, for whose completion digital media, with their specific potential, are necessary.

R - *Redefinition*: Teaching is designed in such a way that learners are faced with fundamentally novel content-related demands that are accessible only through digital media.

In mathematics lessons, it must be decided by the teachers on a case-by-case basis with regard to pedagogical and didactic goals whether and how the digital media can be used for the development of students' competencies.

Digital competencies

The "European Framework for the Digital Competence of Educators" (DigCompEdu) (Redecker, 2017) provides a framework "to promote the digital competence of their citizens and boost innovation in education" (EU, 2017, Foreword). DigCompEdu emphasises six main areas of the professional activities of educators:

- Area 1: *Professional Engagement*. Using digital technologies for communication, collaboration, and professional development.

- Area 2: *Digital Resources*. Selecting, creating and sharing digital resources.
- Area 3: *Teaching and Learning*. Managing and orchestrating the use of digital technologies in teaching and learning.
- Area 4: *Assessment*. Using digital technologies and strategies to enhance assessment.
- Area 5: *Empowering Learners*. Using digital technologies to enhance inclusion, personalisation and learners' active engagement.
- Area 6: *Facilitating Learners' Digital Competence*. Enabling learners to creatively and responsibly use digital technologies for information, communication, content creation, wellbeing and problem-solving.

One aim of these activities, especially in the field of digital problem solving, is "to incorporate learning activities, assignments and assessments which require learners to identify and solve technical problems, or to transfer technological knowledge creatively to new situations" (p. 23). However, this transfer also provides the teachers with feedback "to create new, original and relevant content and knowledge" (p. 82). Note that these digital competencies are for educators in general, not specific to mathematics educators. Next, we focus on competencies specific to mathematics teachers. We bring two conceptualisations of such specifications, an earlier one by Niss and Højgaad (2011) and a more recent one by Geraniou et al (2024).

Teaching competencies based on the KOM Flower

After defining competence for mathematics students (Niss & Højgaad, 2011), it was a natural evolution for the authors to define "What mathematical competencies does a good mathematics teacher ought to have?" (p. 84). Importantly, they stress that a first and unquestionable condition is that the teacher must be fluent and knowledgeable about mathematics. The task of mathematics teacher training is to develop the following didactic and pedagogical competencies:

- Curriculum competency – being able to evaluate and draw up curricula. That is, a teacher needs to be able to evaluate the curriculum, and draw up on it the implication for planning her teaching.
- Teaching competency – being able to think out, plan and carry out teaching.
- Competency of revealing learning – being able to reveal and interpret students' learning.
- Assessment competency – being able to reveal, evaluate and characterise the students' mathematical yield and competencies.
- Cooperation competency – being able to cooperate with colleagues and others regarding teaching and its boundary conditions
- Professional development competency – being able to develop one's competency as a mathematics teacher.

The development of these competencies is a long-standing, or even a lifelong process. The starting point is the preservice teacher education, and it is an on-going process during in-service teachers carriers. Discussing and reflecting on one's own practical work in the classroom is important, preferably in a collegial team at the respective school, and it should be regularly updated in series of professional development.

This competence model has been updated in different follow-up considerations. One recent model, integrating some of the most important competence models, refers specially to mathematics teaching competencies with Digital Media.

Digital competencies in the context of mathematics education – MDCT-Model

Geraniou et al. (2024) proposed a model of "mathematical digital competency for teacher (MDCT)". Based on their conceptual notion relevant for mathematics students, "mathematical digital competency (MDC)" (Geraniou & Jankvist, 2019), the DigCompEdu framework (Redecker, 2017), the concept of instrumental orchestration (Drijvers et al., 2020; Trouche, 2005), the documentational approach to didactics (Gueudet & Trouche, 2009) and the KOM mathematical competencies framework (Niss & Højgaard, 2011), they proposed three characteristics of MDCT:

> [MDCT1] Being able to engage in a techno-mathemat-ical discourse at a pedagogic and a meta-pedagogic level. *This means being able to orchestrate students' engagements in a techno-mathematical discourse as well as incorporating new resources and techno-mathematical development into well-known teaching situations building on the resource-document and scheme-technique dualities. In particular, the didactico-pedagogical competencies of teaching and assessment are relevant in this regard* (Geraniou et al., 2024, p. 631).

This refers to teachers' ability to orchestrate students' ideas, beliefs, working styles and thinking of mathematical objects, represented in digital representations, inside of mathematics as well as environmental problem solving situations, and to develop the understanding of the relation between mathematical objects and their representation.

> Example 1. The aim of a teaching unit is to understand a parabola as a geometric object: Given are a straight line g and a point $P \notin g$. A parabola is the set of all points that are equidistant from P and g. This concept could be approached using a real instrument for drawing parabolas or a dynamic geometry system (DGS) simulation of this instrument. The design of the teaching unit must consider the relationship between the practical activity and the digital simulation, the transition to the symbolic level when determining an equation for the parabola, and the interrelationship between the geometric and symbolic representation.

This example shows that the content knowledge about parabolas, the pedagogical knowledge about adequate teaching strategies and the technological knowledge about possible resources in relation to the problem solution are the basis for this teaching unit. Beyond knowledge, it is necessary to plan, to develop, to deliver and to reflect the unit and the teaching methods. Moreover, the goal of this unit, the development of understanding and students' competencies are the means of regulations for this unit.

> [MDCT2] Being aware of which digital tools to apply within different mathematical situations and context and *being* aware of the different tools' capabilities and limitations, so as to think, and act, pedagogically with these tools, while considering the benefits and limitations of these. *This means being able to engage in a process of documentational genesis, i.e., the instrumentation-instrumentalization duality, which brings resources into the classroom as well as orchestrating students' explorations of affordances and limitations. In particular the didactico-pedagogical competency of curriculum plays a central role here* (Geraniou et al., 2024, p. 631).

This refers to teachers' ability to choose the adequate digital tool for special problem solving situations, knowing about the possibilities and constraints of this tool and to design and implement a lesson for students' use of this tool in this situation.

Example 2. To determine the (first) derivative of a function at a point P on the graph of this function – graphically, numerically and symbolically, one must consider the dynamic change of a secant through P and another point on the graph, recognise the limits of this visualisation, make the transition to the numerical level, preferably using a spreadsheet, and finally, the limit process of a sequence of points approaches a fixed point. This example illustrates the interrelationship between knowledge of the limit process and dynamic activities at the graphical, numerical and symbolic levels. It also demonstrates the understanding of pros and cons of DGS vs. spreadsheet.

The content knowledge about the derivative of function in a point and the different steps of the development of this concept is the basis for the choice of the digital tools, of the adequate representations of these tools, and of the adequate sequencing of these tools and their representations. Even more important is clarifying the limits and constraints of the tools concerning the development or construction of the mental concept of derivative in the heads of the learners.

> [MDCT3] Being able to use digital technology reflectively in problem solving and when doing (learning or teaching) mathematics. *This means being able to orchestrate students' approaches to problem-solving, involving a scheme-technique duality, where the digital resources allow students to connect instrumented techniques to mathematical schemes. Besides the teaching competency, in particular the didactico-pedagogical competency of uncovering learning plays a central role for this type of MDCT* (Geraniou et al., 2024, p. 632).

This refers to the "teachers' need to develop new interdisciplinary pedagogical approaches to integrate PCT [Programming and Computational Thinking] and mathematics" (p. 632). With the knowledge of different models of the development of concept formation teachers are able to observe and to reflect students' actions during the problem solving process, especially the use of digital media, and to decide on the adequate feedback and support concerning the on-going process.

Example 3. When learning about the altitude of a triangle in a DGS environment and starting with an acute-angled triangle, moving a triangle vertex may lead to the case where the altitude coincides with a side of the triangle and even 'disappears' if the triangle becomes obtuse-angled: the altitude is 'moved' outside the triangle. These two situations might be unexpected to the students. For the teacher, this could be a means to uncover the misconception 'the altitude of a triangle is inside the triangle' and make it a learning opportunity, leading to the observation of the location of the intersection of the lines along the altitudes in dependence of the shape of a triangle.

This example shows the interrelationship between students' actions and teachers' reactions. Teachers have to have the competence to recognize students' knowledge, especially misconceptions, about the intended goal of the activity and to lead and structure the on-going problem solving process. It is also necessary to react to unexpected students' actions and to spontaneously vary the planned teaching strategy.

Knowledge models developments – discussion

To summarise, we see in the definition of Niss and Højgaard – "*A mathematical competency* is someone's insightful readiness to act appropriately in response to *a specific sort* of mathematical *challenge* in given situations" (2019, p. 12) – a valuable description of teachers' competence. We interpret the term readiness with the orientation towards action, acting insightfully and openness for challenges before, during and after teaching processes. These are three main guidelines for teachers beyond knowledge. Knowledge is the basis for all these actions. Content, pedagogical and technological knowledge and determining the teaching goals are the prerequisites for the development of teachers' competencies concerning the teaching process. The kind of digital media and the level of integrating them in the sense of the SAMR model must be decided in relation to the teaching content, the knowledge of the learners and the environment teaching takes place. The digital competencies in the DigCompEdu framework and the teaching competencies in the KOM Flower serve as overarching guidelines for the development of teachers' competencies, the MDCT model gives criteria for the development and the evaluation of specific activities with digital media. To develop teachers' competencies in relation or beyond knowledge is a challenging task for teacher education, for teachers' professional development and should always be seen in relation to the daily work of teachers in the classroom.

Questions for Discussion:

(1) Do knowledge-based models alone sufficiently capture the factors that influence teachers' effective integration of technology, or is it necessary to incorporate other constructs?

(2) How can future models integrate non-knowledge dimensions without losing analytical clarity or theoretical coherence?

(3) What are the implications of expanded models – including knowledge and other dimensions – for research, teacher education, and the design of interventions to support effective technology integration?

CONCLUSION

This RF starts with the comparison of the existing models of teachers' knowledge specific to the use of digital technology – TPACK, TKMT, and PTK. This comparison highlights both convergences and tensions in current attempts to conceptualize mathematics teachers' professional knowledge in technology-rich environments. While these models offer valuable lenses for examining the interplay between mathematical, pedagogical, and technological dimensions of teaching, ongoing developments toward framing teachers' digital competencies signal the need to extend their scope. In particular, the emergence of AI-based tools challenges existing assumptions about the nature of mathematical activity, the distribution of agency between teachers, students, and technologies, and the forms of expertise required for meaningful classroom orchestration. Looking ahead, refining and possibly reconfiguring established knowledge models will be essential to capture how teachers learn to interpret, adapt, and critically evaluate AI-mediated mathematical practices. Such work calls for sustained theoretical dialogue and empirical inquiry across contexts and traditions. In this perspective, the present Research Forum is intended to spark discussion and stimulate further research on these issues, contributing to the collective effort to rethink mathematics teacher knowledge in times of rapid technological change.

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