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Presenting multiple representations at the chalkboard: bansho analysis of a Japanese mathematics classroom

Shirley Tan[a], Stéphane Clivaz[b] and Masanobu Sakamoto[c]

[a]International Centre for Lesson Studies, Graduate School of Education and Human Development, Nagoya University, Nagoya, Japan; [b]Mathematics and Science Education Department, Lausanne University of Teacher Education, Lausanne, Switzerland; [c]Graduate School of Education and Human Development, Nagoya University, Nagoya, Japan

ABSTRACT

Pupils' ability to represent mathematical concepts in multiple ways is a central aspect of mathematical competence and communication. Thus, classroom instructions should be able to support learners in using multiple representations (MRs) to increase the quality and quantity of connections to a network of ideas. Given the importance of bansho (board writing and organisation) in Japanese mathematics classrooms, this study aimed to investigate how MRs are presented as bansho in a mathematics classroom. Guided by a coding scheme of MRs on bansho content, the analysis revealed the ways the MRs are facilitating (or hindering) pupils' understanding. In considering the effect of the sequence and translation of MRs identified in this study, it is important to focus on these aspects of lesson design in the future. The relevant findings are also crucial to illustrate to the educators and researchers how to explore the processes involved in the use of MRs and the critical factor that contribute to the success/failure of such processes through a detailed examination of the bansho.

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Introduction

The concept of representations is widely discussed in mathematics education because it is deemed useful in supporting mathematical thinking. In addition, how representations are dealt with is also acknowledged as one of the key quality aspects of interaction processes in the mathematics classroom (Ainsworth, Bibby, and Wood 2002; Duval 2006; Dreher, Kuntze, and Lerman 2015). Parallel to that, pupils' ability to handle representations and to change between representations is considered a core element of mathematical competence because mathematical concepts can only be accessed through representations, therefore making it central for the construction process of the pupils' conceptual understanding (Goldin and Shteingold 2001; Duval 2006). Furthermore, the ability to represent mathematical objects in multiple ways is perceived as a core element of mathematical competence and mathematical communication (Kuntze et al. 2018). Therefore, mathematics teaching and learning should support learners to use flexibly multiple representations (MR).

CONTACT Stéphane Clivaz stephane.clivaz@hepl.ch

The medium MRs are manifested and presented differ depending on many factors. In a classroom context where board-writing is emphasised, it is reasonable to assume that the chalkboard is employed as the primary medium of teaching and learning. Japan is one of the countries well known for its board-writing, with a unique term coined for such usage: bansho. Particularly in a Japanese mathematics classroom, bansho is employed primarily to represent mathematical concepts because 'writing and the development of representational techniques are indispensable for doing and thinking mathematics' (Greiffenhagen 2014, 505). Various research has concluded that bansho is one of the significant characteristics in Japanese mathematics classrooms (e.g. Shimizu 1999; Yoshida 2022; Takahashi 2006; Tokyo Gakugei University 2014; Tan et al. 2021).

While there has been extensive research on MRs and bansho, respectively, the exploration of the MRs on bansho and their interactions with pupils' understanding is an area that is yet to be studied, nonetheless is worth explicating. The process of improving teaching and learning should consider the impact of bansho in an actual classroom, particularly how bansho can be used to facilitate (or hinder) pupils' understanding. The necessity of such exploration is based on the findings of previous work, which illustrates that, while MRs can be helpful in pupil's learning, it has also been cautioned that 'MRs may fail to enhance students' learning if they are not used in the 'right' way' (Rau and Matthews 2017, 531). Notably, for representations to be effective, pupils must correctly interpret each representation and make connections among MRs. If these conditions are not met, the use of MRs may hinder instead of facilitate pupils' learning (Rau and Matthews 2017). Indeed, in a Japanese mathematics classroom where bansho is used as the primary means of content visualisation, the way MRs are presented and dealt with could provide more information on pupils' learning processes.

Research objectives

The study aimed to investigate how MRs are presented as bansho in a Japanese mathematics classroom. Subsequently, the ways these representations are facilitating (or hindering) pupils' understanding were also examined.

Conceptual framework

To investigate how MRs are presented as bansho in a mathematics classroom, a framework that could help to identify aspects of multi-representational design and forms of MRs is necessary. Thus, Ainsworth's Design, Function, Tasks (DeFT) framework (Ainsworth 2006) was chosen. It is a framework used to guide the design and applications of dimensions in multi-representational systems, and it is developed by reviewing a vast range of literature in cognitive psychology/science, education, artificial intelligence, and curriculum studies (Ainsworth 2006). Specifically, the current study centres on the design parameters of the framework because it allows the authors to describe in more detail the aspects of MRs included in the lesson and might reveal the pedagogical functions these aspects are playing. Five aspects are addressed in Ainsworth's design parameters: number, information, form, sequence, and translation. Contrary to Ainsworth's study that aimed to investigate how redundancy in information is distributed in each representation is reduced while students gain expertise in their learning, our study does not aim to address such issues. Therefore, we have decided to exclude the information aspect, which focuses on how information is distributed over the MRs and the complexity of MRs.

Apart from excluding the information aspect from the DeFT framework, we have also defined the form aspect of the DeFT framework more precisely. Form refers to the modality aspects of representations and has received the most attention in learning design because 'it strongly impacts upon learning processes and outcomes' (Ainsworth 2006, 193). Thus, a framework to be the reference standard for exploring the form of MRs in this study is necessary. For this purpose, we have selected the framework of representational modes in mathematics education by Nakahara (1995), which was formed based on Bruner's Enactive-Iconic-Symbolic (EIS) principle and Lesh's representational system (Nakahara 1995). In Nakahara's framework, representational modes are divided into five categories as below, and they move from lower to a higher level of abstraction:

(1) Realistic representation: representations based on actual states and objects
(2) Manipulative representation: representations that have been artificially fabricated to supplement the dynamic operation of objects
(3) Illustrative representation: representations that use illustrations, figures or graphs
(4) Linguistic representation: representations that use everyday languages
(5) Symbolic representation: representations used in mathematical notations

The decision to use Nakahara's representational modes was justified because this framework has a more detailed categorisation than Bruner's EIS principle, where Nakahara further divided 'enactive representation' into 'realistic representation' and 'manipulative representation'. In addition, Nakahara also made a distinction between 'linguistic representation' and 'symbolic representation' as an extension of Lesh's 'written symbols'. With such modifications in Japanese originated Nakahara's framework, it is deemed that those categories in the framework can capture the mathematical representations of the Japanese lesson in this study.

That said, the conceptual framework of this study is an adaption of two frameworks (see Table 1), namely the DeFT framework and representational modes in mathematics education. The latter is explicitly used as an elaboration for the form dimension of the DeFT framework.

Table 1. Conceptual framework of the study.

Dimension	Definition
Number	The number of MR present
Form	The modalities of the MRs and defined in this study as: (1) Realistic representation (2) Manipulative representation (3) Illustrative representation (4) Linguistic representation (5) Symbolic representation
Sequence	The order of MRs
Translation	The degree of support provided to move between MRs

Methods

This study intends to emphasise the process and meaning, which is inaccessible through a quantitative study. Therefore, this paper adopts a qualitative method of case study analysis to reveal how all the parts work together to form a whole (Merriam 1998). In doing so, we intend to take a stand, as advocated by Patton (1985 as cited in Merriam 1998). of not typifying a lesson but to understand the phenomenon in its unique natural context and the interactions in the phenomenon. Although bansho observed in this study shares common characteristics of bansho in Japanese mathematics classrooms, such as helping students to see connections between different parts of the lesson, comparing, contrasting and discussing ideas that students present and organising students' thinking (Takahashi 2006; Yoshida 2022) the Japanese mathematics lesson analysed in this study does not necessarily represent a typical lesson in this country. It is, however, worth noting that the lesson observed in the study consists of four pupils from an ordinary primary school in Japan. The teacher has more than 15 years of teaching experience. We are well aware that the number of pupils is relatively small, and we perceive it as an advantage as it allows for a more in-depth observation and analysis, compared to an average Japanese class size of 27.2 pupils (OECD 2021).

Data collection

The data site for the study was a classroom with four pupils in a rural primary school in Aichi Prefecture, Japan. The consent-seeking confirmation from the school was obtained after presenting the participants with the research purpose and the scope of data usage. Assurances regarding the anonymity of participants and confidentiality of the data collected were also given to all the participants, both teachers and pupils.

The lesson was recorded with two video cameras and two audio recorders. The video camera fixed at the back of the classroom was set to record the process of bansho formation. One digital camera was used to capture pupils' learning materials, and the researcher's field notes were also used for data collection. Particular attention was paid to pupils' utterances and how these were reflected on the chalkboard. Then, the bansho formation process (what/how/when pupils' utterances are written on the chalkboard) was reproduced.

The topic of the lesson under observation was 'What is the hidden number?' with one *hatsumon* (key question for provoking pupils' thinking). The question was presented at the beginning of the lesson in a *hanashi* (story). Its English translation is presented below:

> *In the beginning, 24 children were playing. Then their friends came. That makes it 35 people altogether.*

Pupils presented solutions to answer the question 'what is the hidden number?' and the solutions were in different representations. All of them were recorded on the chalkboard by the teacher and by direct pupils' participation where pupils wrote on the chalkboard. The bansho at the end of the lesson is as in Figure 1 and the English translation of the bansho is shown in Figure 2.

Japanese language was the medium of instruction and therefore, the data collected were in Japanese language. The principal data were the video recording of the lesson and the lesson transcript. The latter was translated into English to facilitate discussion among the

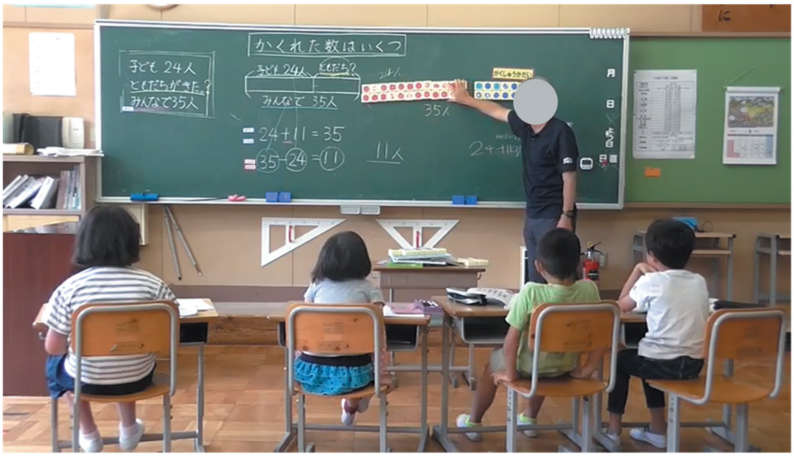


Figure 1. Actual bansho at the end of the lesson.

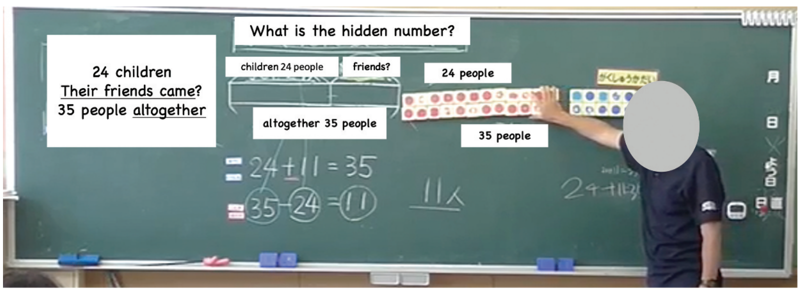


Figure 2. Actual bansho at the end of the lesson (Translated into English).

authors. Therefore, the data analysis was conducted in both Japanese and English, as the second author has an elementary level of Japanese language which is not sufficient for academic purpose. The excerpts of data included in this article are the same English translation used by the second author, and for common discussion among the three authors.

Data analysis

The video and audio recordings were used to produce the lesson transcript. Subsequently, the bansho formation process was reproduced to help us identify four types of crucial information: i) who (the utterer and the writer), ii) what (the bansho content), iii) the how (the way the content was presented and iv) when (the sequence of the bansho content). Then, the lesson was divided into several segments or parts, a method advocated in transcript-based lesson analysis (TBLA) to better understand the segments' relationships (Matoba 2017). The segments are presented in Table 2.

Next, a coding scheme was devised to understand better the relationships among the bansho content and the utterances, focusing mainly on the MRs. The coding scheme is a result of the synthesis of two primary references from the literature. The first reference was Nakahara's representational modes in mathematics education, and it is reflected in

Table 2. Segments of the lesson.

Segment	Time (minute:seconds)	Utterance	Content
1	0:46-2:54	1-27	Confirmation of lesson content
2	2:55-6:54	28-71	Problem and issues to think about in the lesson
3	6:55-10:36	72-132	Pupils working on the problem in their heads
4	10:37-14:28	133-188	Drawing of the tape diagram
5	14:29-18:37	189-267	Thinking time and presentation of ideas
6	18:38-20:49	268-288	Pupil B crying
7	20:50-21:26	289-306	Pupil B's presentation
8	21:27-35:25	307-396	Pupils' explanations of their solutions and ideas
9	35:26-47:37	397-464	Verification of solutions to convince pupil D of his 'puzzle'
10	47:38-52:57	465-493	Teacher bringing in the number-figure blocks

the dimension of the 'form of representation' of the coding scheme. This framework is also integrated into the conceptual framework of the study. Its reliability and coherence with the study design are deemed to be able to yield categories that are meaningful and relatively discernible. Such features are essential in devising a coding scheme that is 'effective from a reliability perspective and efficient from a resource perspective' (Garrison et al. 2006, 2). Apart from a framework that could examine mathematical representations, a framework that categorises bansho content and action is necessary. Therefore, the second reference is adapted from Tan's coding scheme of bansho choreography and bansho transition (Tan 2021). Her coding scheme, which consists of 11 categories and three main elements of bansho (teacher's instruction, pupil's idea, supplementary object), has been utilised to analyse bansho in Japanese schools. In our study, we have added three actions performed on bansho content, namely 'gesture', 'removal', and 'position', as they were observed to occur frequently in the lesson. Thus, they are reckoned to be significant for the analysis.

There are three main dimensions in the coding scheme, namely form of representation, motion, and doer. The first element contains categories of MRs on bansho. The second element deals with the motions of bansho content or bansho-related content. It focuses on the content recorded as bansho or actions performed on the content. Finally, the third element, doer, is designed to indicate the writer and the utterer of the bansho content. For instance, Teacher-Pupil should be understood as the teacher writing on the board with the content originating from the pupil's utterance. Each category is assigned a code. The doer, however, is ascribed with a colour code as well, due to the limitation to represent all three elements in a diagram. Therefore, the colour code will represent each category in the doer element, as illustrated in Table 3 and Figure 4.

Each bansho action was coded using this coding scheme, and the concept map of the research method is presented in Figure 3. As illustrated, each MR is coded with the dimensions from the coding scheme (form, motion, doer), with the sequence, translation and number of MRs taken into consideration.

Findings

Every bansho action recorded in the lesson was coded using the coding scheme. An excerpt of the results is included in Figure 4 below:

Table 3. Coding scheme for MRs on bansho.

Dimension	Category	Sub-category	Code/Colour
Form of representation	Realistic	-	RR
	Manipulative	-	RM
	Illustrative	-	RI
	Linguistic	-	RL
	Symbolic	-	RS
	None	-	NR
Motion	Text	-	FT
	Image	-	FI
	Gesture	Point	FGP
		Slide	FGS
		Count	FGC
		(un)Cover	FGV
	Highlight	Emphasise	FHE
		Connect	FHC
	Attachment	Paper strip	FAP
		Nameplate	FAN
		Learning material	FAL
	Position	Paper strip	FPP
		Nameplate	FPN
		Learning material	FPL
	Removal	-	FR
	None	-	NF
Doer	None	-	NF
	Teacher-Teacher	-	TT
	Teacher-Pupil	-	TP
	Pupil-Pupil	-	PP
	Pupil-Teacher	-	PT
	None	-	NP

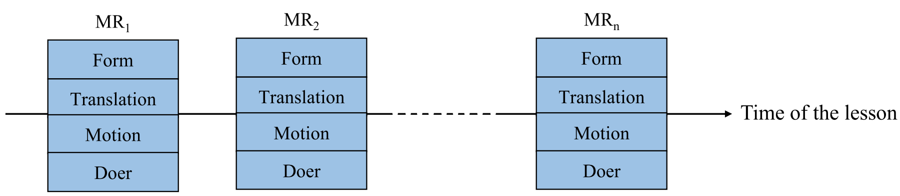


Figure 3. Concept map of the research method.

Altogether there were 145 bansho actions recorded and therefore coded as presented in Table 2. The table contains five different types of information: i) segment of lesson ii) time of lesson iii) form of representation iv) motion v) doer(s). For instance, the bansho action at 08:38, which belongs to Segment 3 of the lesson, involves pointing (FGP) to a linguistic representation (RL) by the teacher while talking (TT, blue).

The results will be discussed in each subheading below, guided by the coding scheme and the conceptual framework of the study. Firstly, the representations on bansho will be discussed in terms of their number, form, sequence, and translation. Then, the relationship between the forms and motions will be explicated.

Segment	Time	Text (FT)	Image (FI)	Gesture				Highlight		Attachment			Reposition			Removal (FR)	No form (NF)
				Point (FGP)	Slide (FGS)	Count (FGC)	Cover (FGV)	Emphasise (FHE)	Connect (FHC)	Paper Strip (FAP)	Nameplate (FAN)	Learning Material (FAL)	Paper Strip (FPP)	Nameplate (FPN)	Learning Material (FPL)		
	01:08												NR				
1	01:11	NR															
	02:03							NR									
	04:31	RL															
	04:50	RL															
	05:06	RL															
2	05:13															NR	
	05:26	RL															
	05:40							NR									
	06:42							NR									
	06:48	RL															
	07:39		RI														
	07:55	RI															
	08:38			RL													
	09:07		RI														
3	09:21	RI															
	09:52			RL													
	10:04			RI													
	10:08	RI															
	15:22	RS															
	16:03	RS															
5	16:28										NR						
	16:35			RS													
	16:50										NR						
	17:47										NR						
7	21:04	RS															
	21:48	RS															
	22:08	RS															
	22:28			RI													
	23:04			RS													
	23:42			RS													
	23:55			RS													
	24:30							NR									

Figure 4. Excerpt of the result of coded bansho actions.

Representation

Number of MRs

The data analysis revealed that seven MRs were used in the lesson and presented as bansho. Out of the seven MRs, two are the combination of two different forms of MRs. Even though there is not an ideal number of MRs to be satisfied, the number of MRs observed in the lesson is reckoned to be relatively high, considering there are five people in the classroom. The relatively high number of MRs could imply that the pupils participated actively in the lesson by sharing various solutions to the problem.

Form of MRs

The form aspect of MR has received the most attention due to its effect on learning processes and the pedagogical function. Four forms (RL, RL, RS, RM) and one compound form (RI-RS) were shown on the board. Only one form from Nakahara's framework was not observed: realistic representation and other forms of the MRs are presented in Figure 5. Each form of the MRs will be discussed in the following subsection, along with an explanation of the sequence.

Sequence of MRs

The seven MRs, which belong to five different forms, were presented in the order of RL→ RI1→ RS→ RI-RS1 → RI2→ RI-RS2→ RM. The lesson began with the teacher presenting a contextual problem orally. He asked the pupils to make a note of the important points of

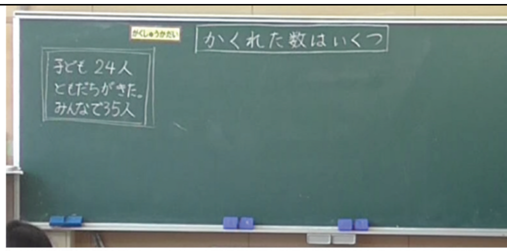
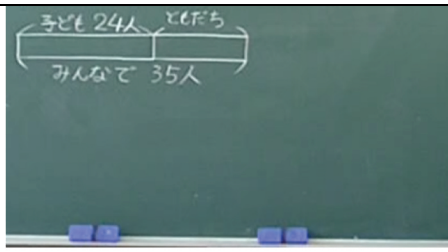
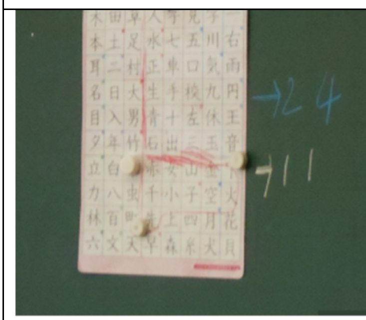
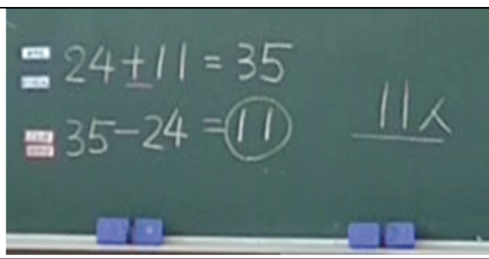
No.	Form of MR	Bansho	Shortened form
1	Linguistic representation (RL)		RL
2	Illustrative representation (RI)		RI1
			RI2
3	Symbolic representation (RS)		RS

Figure 5. Forms of MRs observed in the lesson.

the story. Then, while listening to the descriptions of the story presented by all four pupils, the teacher wrote the story on the chalkboard. This representation is categorised as RL. Then, the teacher drew the tape diagram, which is categorised as RI1, to represent the story problem. The information accompanying the tape diagram was extracted from the story by Pupil B and C, with the guidance of the teacher. Following the tape diagram, the teacher asked the pupils to present their ways of solving the problem of 'how many

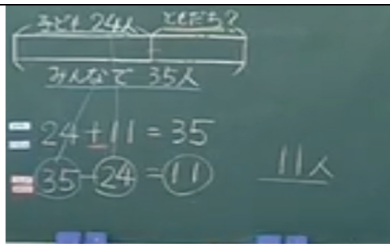
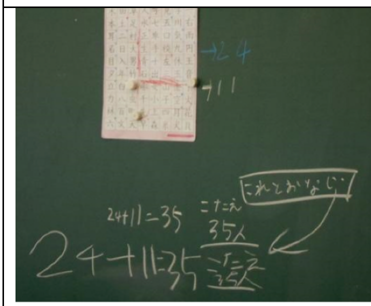
4	Illustrative-Symbolic representation (RI-RS)		RI-RS1
			RI-RS2
5	Manipulative representation (RM)		RM

Figure 5. Forms of MRs observed in the lesson.

friends have come'. Two equations were presented and categorised as RS. The first equation was presented by Pupil C and D, while the second equation was presented by Pupil A and B. Although the pupils could identify the answer to the question, they were having difficulties explaining their reasoning to arrive at the solution: 11. All of them attempted to explain the steps of additional and subtraction operations. It is also worth noting that, Pupil B was the only person who used the tape diagram (RI1) to justify her reasoning and her equation. The teacher noticed the pupils' struggle and facilitated the *neriage* (consensus-building discussion) by drawing lines to connect the corresponding information between the tape diagram and the equation. Since this representation is a combination of RI and RS, it is categorised as RI-RS1. The teacher also tried to share the concern of Pupil D, who was not convinced by the subtraction equation because, to him, the keyword 'altogether' is equivalent to addition. Subsequently, Pupil A, who seemed to have an idea to resolve Pupil D's concern, went to the chalkboard with a kanji grid table, her kanji grid table (a table with adopted logographic Chinese characters used in the Japanese writing system). She framed 25 grids (which should have been 24) and 11 grids of the table to illustrate every single unit. This representation is categorised as RI2. Pupil A continued to explain how 24 and 11 are represented using the kanji table and the equation $24 + 11 = 35$. This representation is categorised as RI-RS2. Inspired by Pupil A's explanation, Pupil C also went to the front and reiterated the same justification. This representation is categorised as RI-RS2 because it comprises the kanji table (RI) and the equation (RS). About 5 minutes before the lesson ended, the teacher realised that some pupils were still not convinced with the subtraction equation. Then, the teacher decided to go and get the number-figure blocks and used them to represent the story problem. The final representation of this lesson is categorised as RM.

Observation of the sequence of the MRs presented as bansho in this lesson has illustrated that the MRs did not progress in increasing order of abstraction level.

Translation of MRs

There are a variety of ways to indicate the connection between representations. In this study, certain support to move between representations was observed. The first way is to use the same numbers, namely 24, 11 and 35, in all the MRs. The whole lesson discussed the same problem; therefore, the numbers used remained the same. The labels, consisting of ‘children’, ‘friends’ and ‘altogether’, were present in the RL and RI1 and omitted in other MRs. However, another cue commonly used to support the translation of MRs, colour, was absent in this lesson. For instance, the two parts of the tape diagram (RI1) were not distinguished with different colours. Even though two colours were in the number-figure blocks (RM), there was no relation signified by these colours between representations. In other words, the colours were only used in this RM. The same tendency was also observed in the kanji table (RI2) used by Pupil A. She wrote the number ‘24’ in blue and ‘11’ in white. However, this distinction of colour was not detected in other representations.

Interactions between form and motion

Upon identification of the MRs presented on the bansho of the lesson, the interactions between the two dimensions of the coding scheme, form, and motion, were explicated. All the bansho actions that have been coded (see Figure 4) are shown in Figure 6. Categories in the coding scheme that were not identified in the bansho actions (i.e. FAP, FAN, FPP, FPN, FR) were omitted from the graph.

From the diagram, several significant characteristics could be noted. Firstly, the high number of FGP in the bansho actions, particularly occurring together with the tape diagram (RI1) and equations (RS). Out of 72 bansho actions coded as FGP, 55 of them were performed on RI1 and RS. Put differently, the teacher and the pupils made explicit references to the tape diagram and equations while speaking. These actions are most often observed in Segment 8 when pupils were explaining their solutions and ideas. The second insight gained from Figure 6 is that the actions of emphasising (FHE) were performed mainly on RI1 and RS. This could signify that the content of the tape diagram and equation is worth paying attention to, so it was often highlighted. In addition, the sliding action (FGS) was performed uniquely on RI1 which could indicate the nature of the tape diagram; it illustrates the relationships among quantities in a problem. Another aspect worth mentioning is the actions performed on the number-figure blocks (RM). Its physical and concrete nature allows learners to manipulate the blocks through hands-on activities. Thus, the observation where the action of counting (FGC), covering (FGV), attaching (FAL) and repositioning (FPL) actions were all conducted only on the RM.

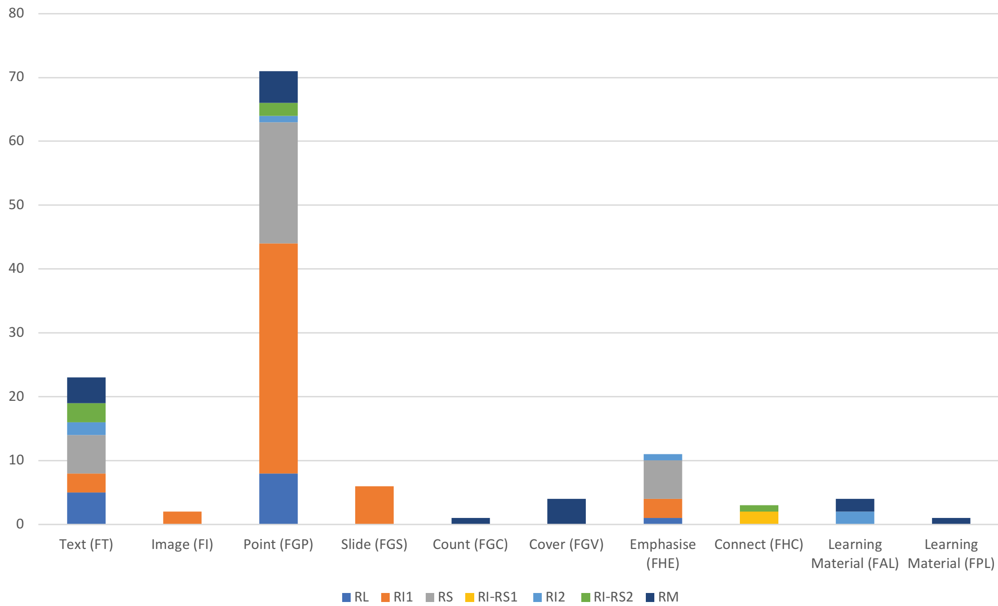


Figure 6. The forms and motions on bansho.

These findings will be elaborated in the discussion section, using data from all three dimensions of the coding scheme: form, motion, and doer.

Discussion

To reiterate the findings, seven MRs which belong to five different forms were identified in the lesson. The sequence of the MRs does not follow a level of increasing abstraction. In terms of the translation of MRs, certain support to move between representations, such as using the same number and labels across some of the MRs, was observed. Based on these findings, two aspects of the learning process will be discussed.

'Arm-wrestling' between the MRs

As observed in the lesson, the teachers and the pupils moved between MRs. While it is not uncommon to move between MRs, there seems to be a pulling of strength in terms of the direction of the MRs. The teacher began the lesson with a story problem, and then he drew the tape diagram to represent the story problem. A tape diagram is a mathematical representation that the pupils have just learnt the day before the lesson. This is evidenced in the exchanges below:

72	Teacher	Yes, so I'll draw a square now. What kind of diagram is this? We had learned it yesterday.
73	Pupil B	Tape diagram.
74	Teacher	Yes, tape diagram. I'm going to draw one now. First of all, I'll draw the first one. The first one. The children.
75	Pupils	44
76	Teacher	40?

(*Continued*)

72	Teacher	Yes, so I'll draw a square now. What kind of diagram is this? We had learned it yesterday.
73	Pupil B	Tape diagram.
74	Teacher	Yes, tape diagram. I'm going to draw one now. First of all, I'll draw the first one. The first one. The children.
75	Pupils	44
76	Teacher	40?
77	Pupil B	24 people.
78	Teacher	Yes, there were 24 people. I'll write that down. When you draw the diagram in your notebook
79	Pupil B	leave a blank line.
80	Teacher	First of all, at the top, please leave a blank line. Below that, 24 people. I'm going to draw that now. Don't make it too small or too big. Medium size will be good.

The fact that the tape diagram is a new representation for the pupils might have led the teacher to focus on the technicalities of drawing the diagram (80 Teacher). Later, when the teacher asked the pupils to share their ways of solving the problem of 'how many friends have come', they presented two equations (RS). When the teacher realised that the pupils were having difficulties explaining their reasoning to arrive at the solution: 11, he provided some guidance by drawing lines to connect the corresponding information between the tape diagram and the equation (RI-RS1). This move was, however, insufficient to facilitate pupils' understanding. The next step taken by Pupil A illustrated the steering of direction by explaining her reasoning with the kanji grid table (RI2). This representation is of a lower level of abstraction and could be perceived as an attempt to go back to a more concrete representation, a 'pre-tape' level with an illustrative linear representation of objects (Murata, 2008). Using the grids of the kanji table, Pupil A was more likely to see every single unit of the quantity represented in the story problem. She further demonstrated how the grids of the kanji table are translated into the equation of $24 + 11 = 35$ (RI-RS2). What is noteworthy is that, prior to RI-RS2, there was already a combination of illustrative and symbolic representation (RI-RS1) provided by the teacher. The pupil's decision to use a specific type of representation, in this case, the kanji grid table, could indicate one's confidence in that particular representation (Hart 1991). When a pupil lacks confidence in interpreting certain representations, he/she tends to avoid using them and opts for a different representation. This might be the case with Pupil A, where instead of using the tape diagram to explain her reasoning, she put forward a new representation that could better illustrate her reasoning. The teacher who noticed the steering of representation by the pupil decided to use the number-figure blocks (RM) and used them to represent the story problem.

In this chain of classroom events, the direction of which representation to be used seems to be determined not only by the teacher but also by the pupils. The teacher aimed for MRs with a higher level of abstraction, but the pupils were not ready to use these MRs in their reasoning. Therefore, the pupils advocated representations more appropriate to their level to accomplish the task. The pupils' actions influenced the teacher's decision; he also proceeded with using a representation of a lower level of abstraction. In this lesson, the sequence of the MRs presented as bansho did not progress in increasing order of abstraction level. Here, two insights could be obtained from this observation. Firstly, the pupils had difficulties using the representations proposed by the teacher because the 'leap' from one representation to another is considered too big. As claimed by Jong et al. (1998), in introducing MRs, its specific order is deemed to be essential because it is beneficial for the learning process. When there is a premature use of MRs, this could provoke a lack of meaning in the learners, which could result in negative consequences for them (Dufour-Janvier, Bednarz, and Belanger 1987). MRs in the lesson appeared to be abstract for the pupils, and when the teacher noticed that, he steered the direction of the MRs to help the pupils construct representations where they feel confident. While 'arm-wrestling' could connote a negative impression when it comes to teaching and learning, the directions of MRs illustrated in this lesson seem to differ. In order to understand and interpret the MRs, the teacher and the pupils were involved in a learning environment where communication and reasoning of the task were emphasised. Such interaction is essential because 'a representation does not represent itself-it needs interpreting; to be interpreted, it needs an interpreter' (Von Glasersfeld 1987, 216). Thus, even when the pupils provided the correct answer to the question, the interpretation of the MRs did not end there. A practice like this could be viewed as an effort to promote the recognition of interpretation, which is an essential part of representations in mathematics learning. The pupils were allowed to learn how representations work and to see the connections among the MRs.

Translating between MRs

As portrayed in the Findings section, a combination of MRs is present in this lesson. With that, it is important to ensure that the learners can use MRs and translate between them. Nonetheless, it is evidenced in many studies that novices often have difficulties flexibly translating a concept from one form of representation to another (e.g. Kwaku, Bossé, and Chandler 2017; Ainsworth 1999; Keig and Rubba 1993). Therefore, adequate support to help students translate between MRs is deemed necessary. In this study, we observed some support for moving between representations. Strikingly, colour, a cue customarily used to support the translation of MRs (Ainsworth 2006), was not observed in the lesson. The absence of colour in the tape diagram (RI1) to distinguish the two parts (children, friends) is intriguing; it is seen as one of the critical factors that contribute to the difficulty of translation. According to Gray et al. (1999), novice students usually use objects that possess colours to translate between representations. However, this support was found missing in this lesson.

The question in the lesson is categorised as an inverse thinking problem, a type of problem that requires students to understand the relationship between addition and subtraction to solve the problem (Goto 2017). For students to solve such problem, the

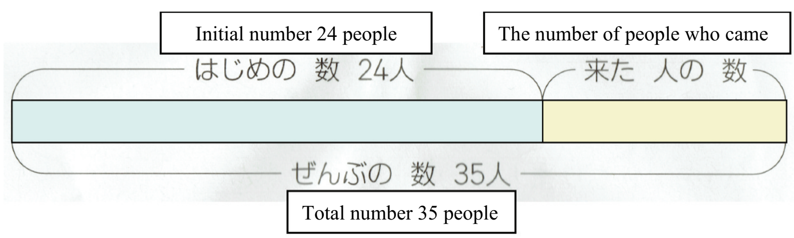


Figure 7. Tape diagram of the problem in the textbook (English translation added).

Ministry of Education, Culture, Sports, Science and Technology (2017) advocated moving from concrete manipulatives to wide line diagrams (tape diagrams) or equations. The tape diagram should correspond with the story problem and consists of all critical information, including those representing the transformation process, i.e. the initial state, the change and the final state (Goto 2017). The usual way to represent this process is through temporal words such as ‘at the beginning’, ‘then’, ‘total’ or through colours. In the lesson, the transformation process was not represented on the tape diagram either way. The tape diagram in the textbook (Shimizu 2014), which the teacher used to prepare the lesson, nonetheless contains both cues: temporal words and colour (see Figure 7).

The absence of two essential elements is seen to be contributing to the pupils’ difficulties in interpreting the tape diagram. Of particular interest is the colour cue, support primarily used to help pupils translate between MRs (e.g. Sakai 2008; Shiraiwa 2014; Wada 2014). An illustrative diagram in mathematical representations plays two chief functions, namely the thinking tool and the explanation tool (Goto 2017). In this case, the tape diagram, which lacks the colour cue, fails to fulfil the role of a thinking tool to help pupils solve the problem by illustrating the quantitative and mathematical relationships and assigning situational meanings to quantities. For pupils to use tape diagrams as a thinking tool, it is necessary to familiarise pupils with the abstract nature and understand the structure of tape diagrams through colours (Ishida 2007).

This observation is also closely linked to the second role of an illustrative diagram, the explanation tool. As discussed above, under the ‘Interactions between form and motion’ heading, a high number of pointing gestures (FGP) was seen to occur together with the tape diagram (RI1). In other words, the teacher and the pupils made explicit references to RI1 while speaking, particularly in Segment 8, when pupils were explaining their solutions and ideas. Apart from FGP, the actions of emphasising (FHE) were also performed on the RI1. Unlike FGP, which originated mainly from the pupils, FHE was performed only by the teacher; he underlined ‘altogether 35 people’ and ‘children 24 people’ and circled the word ‘friends’ while speaking. Crossing the three dimensions of the coding scheme: representation, motion, and doer, it could be deducted that the tape diagram has fulfilled the role of the explanation tool. The pupils used it to explain their solutions to their peers by engaging in their own sense-making process via constant reference to the tape diagram. On the other hand, the teacher used the tape diagram to draw pupils’ attention to the core aspects of the problem. Nonetheless, it appears that the action of emphasising the wordings on the tape diagram was not sufficient. Merely circling or underlining them did not help pupils see how quantities relate to each other and how they have changed. This brings the discussion back to the translation between MRs; the translation could have been improved using colours and temporal words. For example, the part-whole relationship and the transformation process could be distinguished using colours and temporal words on the tape diagram (RI1), like the one shown in the textbook.

The discussion above has drawn our attention to a vital aspect of MRs, translation between MRs. Ainsworth (1999) claimed that the role of translation between MRs is not only crucial in improving pupils’ understanding and retention of mathematical concepts; it also influences the fit between the design and the learning objectives. The decision to use MRs should consider the supports to translate between MRs so that the relations between them are made very explicit.

Implications and conclusion

The study began with the premise that pupils' ability to represent mathematical objects in multiple ways is a central aspect of mathematical competence and mathematical communication. Thus, classroom instructions should be able to support learners in using MRs to increase the quality and quantity of connections to a network of ideas (de Walle et al. 2013). This study has identified the support needed, and the absence of such support could explain why pupils were unsuccessful in using MRs to show their understanding of the mathematical concepts. We believe that the findings may improve knowledge about lesson design where teachers could consider the effects of the sequence and translation of MRs on learning. The detailed account of bansho use in an actual Japanese mathematics classroom presented in the study would also allow teachers to use it as a source of reference, especially for those who might not have observed lessons of such format in their countries. As for teacher educators and researchers, this study could serve as material to be disseminated to in-service or pre-service teachers on how to make learners' thinking visible and the effect of MRs on their mathematical learning. Additionally, the use of textbook excerpts, pictures of bansho and lesson transcript, as illustrated in this study, could model an evidence-based discussion in a professional teacher development setting. For future studies, we intend to apply the framework to more elaborated mathematical problems or larger class sizes.

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