Pre-service teachers' intuitive conception about fractions

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Individuals are familiar with the concept of fractions, but what are their intuitive conceptions? In this exploratory research, we study the intuitive conceptions through two components: the interpretations and the registers of representation. To do this, we submitted an open-ended survey to Swiss pre-service teachers. The results show the prevalence of two interpretations (part-whole and quotient) and two registers of representation (symbolic and visual). This suggests the coexistence of a spontaneous conception of the part-whole associated with one or the other of the registers of representation and knowledge coming from teaching related to quotient interpretation.

Keywords: Fractions, intuitive conception, conceptual development, representation, mathematics education.

Introduction and conceptual framework

Fractions are known to be a difficult topic for both teachers and students, and can be challenging to learn and teach. Research shows several reasons for the difficulties; we detail the main ones below (Pitkethly & Hunting, 1996).

First, because the properties of natural numbers are not always relevant to rational numbers, individuals manipulating rational numbers with the rules of natural numbers will make mistakes or construct incorrect knowledge. Ni & Zhou (2005) called this bias "Whole number bias". This bias expresses a lack of comprehension of the concept involved. This explains why students say that $\frac{1}{4}$ is greater than $\frac{1}{3}$, because 4 is bigger than 3.

Second, individuals have difficulty making connections between different situations involving fractions. Behr et al. (1983) defines five interpretations of fractions: the fraction *part-whole* (number of equally sized parts out of a whole), the fraction *measure* (a point on the number line), the fraction *quotient* (the result of a division), the fraction *ratio* (the relation/ratio between the numerator and denominator) and the fraction *operator* (operating on a number or an object). Each of these interpretations has its own specificities (representations, procedures) which make it more or less easy to solve a problem involving fractions. These interpretations help to understand, among others, the sharing in equal parts (part-whole), the density of rational numbers (measure) or the fraction equivalence (ratio). If some students do not identify or know an interpretation, they might have difficulty solving specific problems. E.g., students who think of the fraction only as a part-whole might have difficulty solving problems involving proportions, which require the fraction as a ratio. If students rely only on one interpretation, this limits their understanding of fractions (Charalambous & Pitta-Pantazi, 2007). The part-whole interpretation is the interpretation that students encounter for the longest, and most frequently, in their mathematics textbooks (Alajmi, 2012).

Third, to represent fractions there are a lot of signs depending on the situation (e.g. words, numbers, or geometric forms. These signs are not the mathematical object itself; they are representations of a mathematical object; they make it accessible. When representations share the same signs, they can be grouped into a semiotic system, called a register. (Duval; 2006) Depending on the register of representation, fractions can be written in different ways. Marmur et al. (2020), based on the research of Duval (2006), defined three registers (see Table 1): the *symbolic register*, the *verbal register,* and the *visual register*. The *symbolic register* includes symbolic writing from the fractional register, the decimal register, and other registers. Also, for the fraction $\frac{1}{2}$, in the *symbolic register* it can be $\frac{1}{2}$ or $\frac{5}{10}$ (fractional register), 0,5 (decimal register), or 50% or 1 ÷ 2 (other writings) (see Table 1a). In the *verbal register*, $\frac{1}{2}$ can be "half", "one for two" (see Table 1b). In the *visual register*, it can be one out of two equal parts (see Table 1c). The visual register is the register that students encounter the most in their textbooks (Charalambous et al., 2010). Adjiage & Pluvinage (2007) also propose the register of the *linear scale*, thus $\frac{1}{2}$ is a point on the number line exactly between 0 and 1 (see Table 1d).

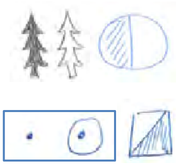
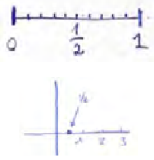
Symbolic register (1a)	Verbal register (1b)	Visual register (1c)	Linear scale register (1d)
Fractional register $\frac{5}{10}$ $\frac{2}{4}$	une demie = a half		
Decimal register 0,5	La moitié = the half		
Other writing 50 %	"un sur deux" = one for two		

Table 1: Registers of semiotic representation for fractions

For Fischbein (1987), students possess knowledge that is easily acquired without instruction and may not correspond to culturally and scientifically accepted notions. This knowledge is referred to as *intuitive conceptions* and can vary significantly from the formal definitions and methods learned in class. Intuitive conceptions include both primary intuitions, which develop in individuals without formal instruction, and secondary intuitions, which emerge when formal knowledge becomes intuitive. Research suggests that intuitive conceptions influence the construction of the concept and can predict and explain a student's performance. If the strategies for solving a task are based on intuitive conception, students will solve it more easily and quickly. On the other hand, if the strategies for solving a task are far from their intuitive conception, students will take longer to solve it and may find difficulties. Therefore, the identification of these intuitive conceptions can provide both an explanation and a prediction for task difficulty (Gvozdic & Sander, 2018). Research shows that intuitive conceptions can be a source of difficulties. For example, in arithmetic word problem solving (Sander, 2018), ninth-grade students were asked to construct division problems with a result that must be higher than the starting values. Almost all students failed. Three quarters of them indicated that it is impossible. They relied on the intuitive conception that the result of the division is always smaller than the initial values and this hindered their reasoning. The influence of intuitive conceptions does not disappear after teaching, as Tirosh & Graeber (1991) showed with pre-service teachers about this same intuitive conception on division. Fischbein (1987) states that teaching new mathematical concepts in school often depends on intuitive conceptions. Gvozdic & Sander (2018) revealed that teachers' comprehension of how children utilize informal strategies was obscured by their own intuitive conceptions of arithmetic operations in the context of solving arithmetic word problems.

The current study

Some studies show that intuitive conceptions are resistant to teaching. They persist and continue to influence individuals even though the concept has been studied at school (Tirosh & Graeber, 1991; see Gvozdic & Sander, 2018 for a review). On the other hand, research considers that students associate familiar knowledge (intuitive conceptions) about part-whole interpretation regarding fractions, but that this knowledge is not connected to the symbolic or verbal register (Mack, 1995). In this study, our goal is to identify the intuitive conceptions of pre-service teachers. For that purpose, it is important to look for what causes resistance to teaching in the field of fractions.

We hypothesize that the part of a whole is the main intuitive conception. To observe this, we investigated conceptions of pre-service teachers. They all have a master's degree and it can be considered that they all have been taught about fractions during their school years. Research already bears on the behavior of teachers and students when they have to solve problems on fractions (Charalambous & Pitta-Pantazi, 2007) but to our knowledge no study has tried to identify the intuitive conceptions. Hence, the research question: *What are the intuitive conceptions of fractions of pre-service teachers?* To get the most spontaneous answers, specific open-ended tasks involving productions were proposed.

Method & Material

Participants

We collected data from 122 participants during the academic year 2022-2023. They are pre-service teachers who are in university training to obtain tenure in the French-speaking part of Switzerland. These pre-service teachers already have a disciplinary master's degree and some already teach. There are 68 women and 54 men studying to be teachers in middle school (54), high school (27) or both (42). They are studying to become subject specialist teachers in French, biology, music for example, and train for one or two subjects. There are 18 participants who train to teach mathematics.

Material

The participants answered a six-question survey. To prevent participants from expanding their answers, they received the tasks in the same order. They had to answer without going back. They were asked to answer as spontaneously as possible and with the first idea that came to mind.

Question 1: According to you, what is a fraction?
Question 2: Give 4 examples of a fraction.
Question 3: Represent in four different ways the fraction $\frac{1}{2}$.
Question 4-5: Give the first problem that comes to your mind for which the solution is $\frac{3}{8}$. Solve it.
Question 6: If you had to explain to somebody else what $\frac{6}{5}$ is, what would you tell him/her?

The surveys were not digital. They were distributed in paper version and the participants had to answer with a pen. This choice was necessary to collect the participants' intuitive conceptions, notably the component of spontaneous representations. Indeed, they had to be able to mobilize some registers of representation (e.g. visual or verbal). However, drawings and mathematical writings are not easy to achieve with a computer. This could have been an obstacle to the collection of these representations.

Tasks 1 and 4-5 are designed to identify interpretations of fractions (Behr et al., 1983). The purpose is to see through their answers which interpretation(s) they will mention, as spontaneously as possible. Tasks 4 and 5 are considered and encoded together.

Tasks 2 and 6 provide additional information on the interpretations of fractions through the mobilization of improper fractions. Task 6 challenges participants on their knowledge about improper fractions. Indeed, in the part-whole interpretation, a continuous quantity or set of discrete objects is divided into parts of equal size. The fraction represents a comparison between the number of parts of the partitioned unit and the total number of parts of the unit. From this point of view, the fraction is always smaller than 1 (Charalambous & Pitta-Pantazi, 2007). In the other interpretations, improper fractions make sense.

Task 3 is designed to identify the registers of representation (Marmur et al., 2020; Adjiage & Pluvinage, 2007) (see Table 1) that the participants spontaneously mobilize. They have to represent the fraction $\frac{1}{2}$ in four different ways. No additional information is given so that the participants can write or draw what they have in mind. It is during the encoding that the registers of representation are identified.

Data analysis

The data are written in a spreadsheet. The interpretations and the register are encoded with the help of indicators. The coding and analysis of the registers of representation, interpretations and mobilisation of the fraction greater than 1 are done independently, they do not overlap.

To analyse the tasks 1, 4 and 5, interpretations are used as indicators: *quotient, operator, measure, part-whole and ratio.* Two more indicators are considered. The indicator *formalised writing* corresponds to the writing of a fractional number as $\frac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{Z}\, et\ b \neq 0$. We encode all responses of this kind in this indicator as the partial answers as $\frac{a}{b} \in \mathbb{Q} \mid a, b \in \mathbb{N}$ for example. The indicator *unclassifiable* corresponds to answers in which it is not possible to identify interpretations or problems that are not problems (e.g., the question of the problem is missing). The answer is encoded with a 1 when there is one of the indicators mentioned. A participant may refer to multiple interpretations in a single response.

For task 2, the fractions are encoded with the following three indicators: *answers between 0 and 1*, *answers greater than 1* and *first answer is greater than 1*. Irrelevant responses (such as drawings) are encoded with a "*na*" (meaning not applicable). For task 6, indicators are: Explanation that $\frac{6}{5}$ is greater than 1, and denial of the fraction greater than 1. The answer is encoded with a 1 when one of the indicators corresponds.

To analyse task 3, the inputs in Table 1 are used: *symbolic register, verbal register, visual register, linear scale register*. We add a *not relevant* category for answers that are not relevant (e.g., a drawing of the Euclidean division). The answer is encoded with the number of times each register was mentioned (maximum 4).

Results

Interpretations of the fractions

To identify which interpretation(s) were mentioned the most (see Table 2), we looked at the number of participants who mentioned it per task (1 and 4-5). In the task bearing on the definition of the fraction (task 1), it is the quotient interpretation that is most mentioned by the participants (57%). In the creation and resolution of a problem for which the result is $\frac{3}{8}$ (tasks 4-5), it is the part-whole interpretation that stands out (49%). Overall, these two interpretations are the most mentioned. The operator (2%) and measure interpretation (1%) are almost absent among the answers. The data of the mathematics pre-service teachers were analysed. Their results are almost identical to those of the whole population of participants, which is why they are not treated independently. In task 1, the quotient is also the most mentioned (61%) and in task 4-5 it is also part-whole (44%).

	Part-whole	Ratio	Operator	Quotient	Measure	Formalized writing	Unclassifiable
Task 1	36%	11%	2%	57%	0%	2%	7%
Tasks 4-5	49%	4%	2%	11%	1%	16%	16%

Table 2: Percentage of participants who mention these interpretations at least once in their answers

The results of task 2 (see Table 3a) show that 36% of the participants propose a fraction greater than 1, and that only 7% of the participants propose it in the first place. 97% of the participants provide at least one fraction between 0 and 1. Furthermore, 44% of the participants propose 4 fractions between 0 and 1. In task 6 (see Table 3b), 5% of the participants deny the existence of the improper fraction (e.g., they interpreted $\frac{6}{5}$ as being the same as $\frac{5}{6}$). 95% of them try to give an explanation. Among these answers, some mention explicitly that the fraction is greater than 1, e.g. "it's a little bit more than 1"; others are less explicit, e.g. "6 : 5 = 1,2".

Task 2	Answers between 0 and 1	97%
	Answers fraction > 1	36%
	First answer is greater than 1	7%

Table 3(a): Percentage of participants who mobilise improper fraction

Task 6	Explanation that $\frac{6}{5} > 1$	95%
	Denial of the fraction >1	5%

Table 3(b): Percentage of participants who accept or deny improper fraction

Registers of representation

To identify the register of representation that comes to mind spontaneously (see Table 4) when it comes to giving four different representations of the fraction $\frac{1}{2}$ (task 3), we looked at the number of participants who mentioned it. 89% of the participants used the symbolic register to represent $\frac{1}{2}$ at least once (66% of the participants used fractional writing, 56% used decimal register), 54% the visual register (54% of the participants used circular surface as representation). Only 9% of the participants mentioned the verbal register and 3% the linear scale register.

Symbolic register	Verbal register	Visual register	Linear scale register	Not relevant
89% including fractional register: 66% including decimal register: 56%	9%	54% including circular surface:46%	3%	11%

Table 4: Percentage of participants and answers who mention these registers at least once

In addition, 33% of the participants only work through the symbolic register (the four answers are in the register); and 8% of the participants treat only in the visual register. The rest of the participants mobilize between two and three different registers. No one mobilizes four registers. For this component too, pre-service teachers in mathematics have almost the same results as the rest of the participants, 33% of them work in one register only, no one mobilizes four registers.

Discussion

The results show that to produce a definition of what a fraction is (task 1), the quotient interpretation is the most used. This interpretation is quite close to the mathematical definition of a rational number and is based on knowledge coming from secondary school. Thus, this intuitive conception would be a secondary intuition (Fischbein, 1987) that derives from formal teaching. In contrast, when it comes to producing a problem about fractions (tasks 4-5), participants tend to mobilize the part-whole interpretation mostly in a context of cake/pizza/pie sharing, where the units (an object or a set of objects) had to be separated into a specified number of identical parts or subsets, and then treated individually. According to Mack's (1995) research, it is informal knowledge, an intuitive conception. For Fischbein (1987), it would be a primary intuition (Fischbein, 1987). Research on students' conceptions of the operation of division reinforces this idea. It reveals that students often overgeneralize principles of natural number operations to fractions, leading them to use a basic partitive approach when interpreting division (Tirosh, 2000). Additionally, the spontaneous mobilization of fractions less than 1 is related to the part-whole interpretation and supports the hypothesis that it is an intuitive conception. Indeed, the part-whole interpretation makes it difficult to represent fractions greater than 1. Finally, even if the fraction greater than 1 is not denied by the participants, some explanations do not allow us to say if it is really accepted; some unclear answers show a discomfort of the participants in dealing with tasks involving these fractions. We can make the hypothesis that the interpretation is not completely mastered because it is not mobilized in the task where they have to justify what an improper fraction is (task 6).

The cohabitation between the intuitive conception and the knowledge resulting from teaching could be reinforced by the division of curricula and the transition between primary and secondary school that Chambris et al. (2017) in the French context describes as a rupture. The curricula of the French-speaking part of Switzerland are comparable to those of France from this point of view.

The measure and operator interpretation are almost non-existent among pre-service teachers. We can hypothesise that they are not part of intuitive conceptions; otherwise, they would be present. They would therefore be the result of knowledge coming from teaching. The non-mobilization of these interpretations by pre-service teachers can be explained by the lack of teaching on these interpretations in primary and secondary schools in the French-speaking part of Switzerland. This could perhaps change in the coming years as the Swiss curricula are changing and fractions will be introduced with the measure interpretation (in 2024 in State of Vaud). This interpretation also allows the passage from the part-whole interpretation to the quotient interpretation (Chambris et al., 2017).

The analysis of the registers of representation revealed that the symbolic register and the visual register are the most present in the participants' answers. These two registers seem to coexist in the participants' mind, one coming from school learning (symbolic register using mathematical signs) and the other coming from familiar knowledge (visual register, for example the sharing of a circular surface representing a cake). On the one hand, 45% of the participants provided all four answers using the same mathematical register, while 44% of the participants gave answers that involved at least two different registers. The remaining 11% provided at least one irrelevant answer. Duval (2006) asserts that to understand a mathematical object, it is necessary to be able to mobilize several of its registers and to be able to make transformations within a register or between different registers. The mathematical object remains the same. He indicates that these transformations can be a source of difficulties. In this study, 45% of the participants did not do so spontaneously, which suggests that they may not have a complete understanding of the concept of fractions. We can hypothesize that participants who did not mention multiple registers may not have a complete mastery of the concept of fractions. To confirm this hypothesis, further investigation is required, specifically by asking more explicit questions regarding these transformations.

The linear scale register is the least mentioned by the participants. However, studies show that this register helps to understand the magnitude of fractions (Siegler & Lortie-Forgues, 2014). Furthermore, intervention studies show that placing fractions on a graduated line can lead to an improvement in the understanding of fractions, particularly for people with mathematics difficulties (Fuchs et al., 2016). The non-mobilization of this register may be due to a lack of the Swiss curriculum. As curricula are evolving and increasingly integrating the measure interpretation and the linear scale register into the textbooks, it would be interesting to re-use this survey with future teachers who will have followed this curriculum.

Conclusion

In order to answer the research question *What are the intuitive conceptions of fractions of pre-service teachers?*, an intuitive conception among teachers would be the part-whole associated with one or the other of the registers of representation: symbolic and visual (circular). This conception coexists with knowledge resulting from teaching: the quotient interpretation.

But then what to do in teaching? Should we start from intuitive conceptions and build on them? Does this risk anchoring these conceptions even more? Or, should we design our teaching to avoid them on purpose? Will the mathematical object make sense to the learner? We can also ask whether these intuitive conceptions are the same among students. This work will also be carried out in the primary school.

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