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Algebraic generalisation of a student with MLD: Skills despite difficulties and implications for diagnostic tests

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Students with Mathematical Learning Disabilities (MLD) are diagnosed in most cases through arithmetic performance tests. The diagnosis identifies them as having severe difficulties in mathematics. For the present study, we investigated the algebraic thinking of a 9th grade student with a diagnosis of dyscalculia through the generalisation of a geometric pattern. Despite her dyscalculia diagnosis, the student was able to solve the proposed problem by developing an arithmetic generalisation and different types of algebraic generalisations. The analysis of the interview with the student shows that, despite the major arithmetic difficulties identified by the diagnosis, the student shows algebraic skills. In addition to showing the skills that students with MLD may have, this result highlights the limitations of the diagnostic tests currently used and the need to supplement them in order to cover other mathematical areas.

Keywords: Algebraic thinking, dyscalculia, generalisation, mathematical learning disabilities, patterns.

Introduction and literature review

Activities involving algebraic thinking are essential for meaningful mathematical learning. In fact, they enable the development of fundamental aspects such as analysing relationships between quantities, modelling, proving, justify, noticing structure and generalisations (Kieran, 2004). Research in mathematics education has long been interested in defining algebraic thinking. Some of the literature focuses on the use of literal symbolism, while others focus on the operations involved. This second approach offers a broad interpretation of algebraic thinking and allows it to be identified in activities that are not restricted to the use of letters.

Researchers have long wondered about the difference between arithmetic and algebraic thinking, which are often interpreted as one in continuity with the other. Radford (2018) identifies three components of algebraic thinking that characterise algebraic thinking:

1. *Indeterminate quantities*. The mathematical situation that enables algebraic thinking includes unknown numbers (variables, unknowns, parameters, generalised numbers, etc.).
2. *Denotation*. The ways of representing these indeterminate quantities and their operations are culturally and historically constructed. It is also possible to use other types of representations rather than alphanumeric symbolism, such as natural language, gestures, etc.
3. *Analycity*. Indeterminate quantities are considered as if they were known numbers or specific numbers and are operated on. Known and unknown numbers are treated in the same way.

In the last decades, we have seen an increase in the attention of political institutions towards Learning Disabilities. Indeed, they are now also part of the official documents of various European countries, which recommend that schools take charge of them: for example, with the 360° Concept (DFJC, 2019) in the canton of Vaud in Switzerland, the context of this paper. In mathematics education, we are seeing an increasing interest in students with Mathematical Learning Disabilities (MLD), and this is also visible from the recent creation of *TGW25 Inclusive Mathematics Education – Challenges for Students with Special Needs* in CERME11 in 2019 (see for example Gregorio, 2022). Learning Disabilities are traditionally associated with a cognitive dysfunction that leads to severe and persistent academic difficulties that are not due to inappropriate pedagogy, sociocultural factors, developmental delay or sensory impairment (APA, 2013). A well-known learning disability is *dyslexia*, which refers to difficulties in accurate and fluent word recognition and poor spelling skills. Concerning mathematics, Learning Disabilities affect number sense, memory for arithmetic facts, accuracy or fluency in calculation and mathematical thinking generically. *Dyscalculia* is another term used to refer to difficulties in processing numerical information, learning arithmetic facts and performing calculations. In addition, students with MLD often present a deficit in working memory, and in particular spatial working memory is impaired, both in sequential tasks and in tasks where visuospatial information had to be maintained simultaneously (Mammarella et al., 2018). Difficulties in visuospatial working memory can have an impact on mathematical abilities as they affect the capacity to work on order and sequence.

MLDs are diagnosed through mathematical performance tests. Most of the tests are on basic arithmetic skills (counting, comparing and estimating collections, transcoding, the four symbolic operations, etc.) and usually with a procedural vision of arithmetic, with the implicit assumption that difficulties in any mathematical domain are due to difficulties in arithmetic (Baccaglini–Frank et al., 2020). This assumption is problematic, because not all difficulties in mathematics are necessarily related to arithmetic. Moreover, it is possible to have difficulties in arithmetic without having difficulties in other mathematical domains. The difference between algebra and arithmetic is therefore particularly relevant when it comes to MLD diagnoses tests, because they have ambitions to test Learning Disabilities in mathematics but are actually very much related to arithmetic.

In this paper I am therefore interested in two issues. Firstly, I want to study the algebraic thinking of a student who has been diagnosed with MLD. I make the hypothesis that a diagnosis of MLD does not necessarily imply that the pupils have difficulties in all mathematical areas and that therefore they may show mature reasoning in algebra. In addition, I explore some implications that these findings might have for the tests used for diagnosis.

Theoretical framework and research question

As mentioned in the previous section, generalisation is a fundamental aspect for algebraic thinking. Radford (2008, 2010) identifies different types of generalisations in the context of geometric patterns (such as the one in Figure 1): arithmetic generalisation, factual generalisation, contextual generalisation, and standard symbolic generalisation. This typology assists the researcher in the identification of certain processes and should not be interpreted in terms of hierarchical stages of biological or cognitive development of pupils.

The relationship between the number of squares and the number of straws in the pattern in Figure 1 can be *generalised arithmetically*. In this case, the relationship between successive terms in a sequence is grasped, yet a formula for any term is not identified and a direct rule for finding the number of sides for any figure is not established. For example, to find the number of sides for step

100, it would be possible to generalise by saying, "you must do 4+3+3+3... etc. and get to 100". This is obviously a generalisation, but it is not algebraic because there is no indeterminate involved. Moreover, it is not analytic because we do not operate on unknown numbers.

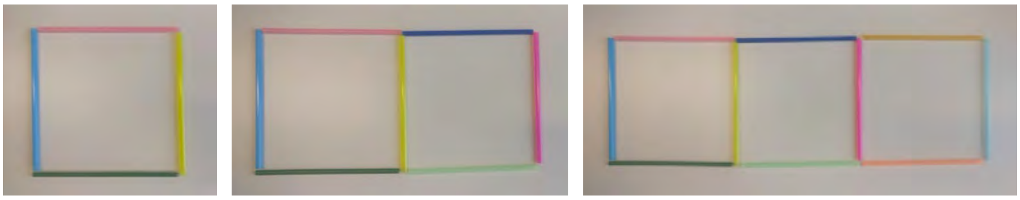

Figure 1: The mathematical problem, a geometrical pattern

Factual generalisation is a first type of algebraic generalisation in which the variable comes into play and one operates on it. At this point, students have grasped the regularity of the pattern and generalised it to any particular figure. They have at their disposal a formula in act composed by actions rather than by formal symbols. The concepts of variable and indeterminate are sensed by the students, but not explicit. For example, one could generalise the pattern in Figure 1 to the case 25 by saying: "you must do 25+25+25+1".

The *contextual generalisation* is another type of algebraic generalisation and concerns a figure that is not specific. However, the resulting formula is still contextual, related to spatial and temporal characteristics and based on the personal interpretation of the pattern. For example, for the pattern in Figure 1 one could say, "the number of the figure for the sides in bottom row and for the sides in top row, plus the number after the number of the figure for the oblique sides". The indeterminate aspect is made explicit and is expressed by the formula "the number of the figure".

The *standard symbolic generalisation* allows detaching from the concrete context. For example, symbols in the formula $3n + 1$ assume a relational meaning in function of other symbols and their narrative dimension disappears.

Studying the algebraic thinking of pupils with MLD is particularly interesting because there are still not many studies on this subject as most focus on arithmetic tasks (Baccaglini-Frank et al., 2020). In addition, their visuospatial memory can be impaired (Mammarella et al., 2018), potentially impacting the order and sequence aspects of tasks such as the generalisation of algebraic patterns. Given these characteristics of pupils with MLD, it is pertinent to ask whether or not they perform as well as pupils without MLD, for whom there is an extensive literature (i.e. Radford, 2008). To study the algebraic thinking of pupils with MLD, the question driving this paper is: *what types of generalisations are used by a student with MLD when studying a geometric pattern?*

Method

Context and participant

The data presented in this paper are part of a more extensive research in terms of pupils involved and proposed mathematical tasks. Data were collected in the canton of Vaud, in the French-speaking part of Switzerland[1]. The region has seen a growing interest in inclusion in schools in recent years (DFJC, 2019). Despite this growing interest, school organisations are divided into different types of schools (ordinary and special schools) and school levels, depending on pupils' grades.

In the following pages, I will focus on the case of Ambre, a pupil diagnosed with MLD and identified as being in severe difficulties in mathematics by her teachers. Ambre is enrolled in 9th grade in an ordinary school, in the level of students with low grades. She has a diagnosis of dyscalculia and dyslexia, she has great difficulties in mathematics, and she is excused from German and English classes in order to lighten her workload. The student is recognised, according to the teacher and the student herself, as being in great difficulty in mathematics, regardless of the subject. Ambre has not yet encountered the chapter about algebraic literal calculation in her schooling.

Procedure

Data were collected through clinical interviews. These are semi-directive and open-ended interviews between the researcher and the student (Ginsburg, 1981). The interviewee solves the assigned mathematical task by making explicit the procedure used. The researcher guides the interview by proposing the task, asking for more detailed explanations where necessary and unblocking the situation with questions when needed. The goal of the interview is to stimulate the student's responses in a way that encourages showing mathematical reasoning. The researcher then sought to create the ideal contextual conditions for this by using the student's claims to allow them to show the full potential of their reasoning, but without replacing them in finding the final answer.

The interviews were filmed and transcribed. The content of the analysis involved the transcribed students' interventions and their written productions. These were analysed using Radford's types of generalisation presented earlier.

Mathematical problem

The interview was about generalising the pattern in Figure 1, a classical problem, where the squares in the picture were made up of straws. The instructions for the problem were as follows. *Here are the first three steps of a sequence of squares. How many straws are needed to form a sequence of 4 squares? And 5 squares? How many straws are needed to make a sequence of 12 squares? How many straws are needed to make a sequence of 100 squares? If you knew the number of squares, could you still find the number of straws? If so, how?*

The task was given in pencil paper, but real straws were proposed to the students in case they had the need to represent the sequence in a more concrete way.

Results

During the interview, Ambre proposes a factual generalisation for 4 and 5 squares by identifying the regularity of the pattern, as we can read in the following excerpt.

Researcher: Okay. So how many straws for 4 squares?

[1] Interview excerpts and the text of the mathematical problem have been translated into English for this paper from the original French version.

. . .

Ambre: We have 4 straws at the top, then 4 straws at the bottom. Then next, we have in the intersections, and we have 3 intersections, those, there [*pointing to the vertical straws in the third picture in Figure 1, but not the first one*]. Plus the 2 at the extremities, so that's plus 2. So, in total, for everything, there, it's equal to 13, so it's 13 straws [*she writes the calculation in Figure 2, top left*].

Researcher: Perfect. Very good. What about the 5 squares?

Ambre: Ah, for 5 squares we do the same thing, but for the 5. So, there will be 5 on top, 4 in the middle and always 2 on the side. So, it will be 16 [*she writes the calculation in Figure 2, bottom left*].

The formula in act that the student proposes is a generalisation that she would be able to apply to any particular number. In fact, for 5 squares she says, “we do the same thing”, and afterwards she reproposes the same strategy for 100 squares (Figure 2, in the middle). Ambre's procedure is noteworthy because, for such small numbers of squares, students typically draw and count (Radford, 2010). Ambre, on the other hand, despite –or thanks to– her difficulties in mathematics (or arithmetic?), relies on the structure of the pattern, highlighting its regularity.

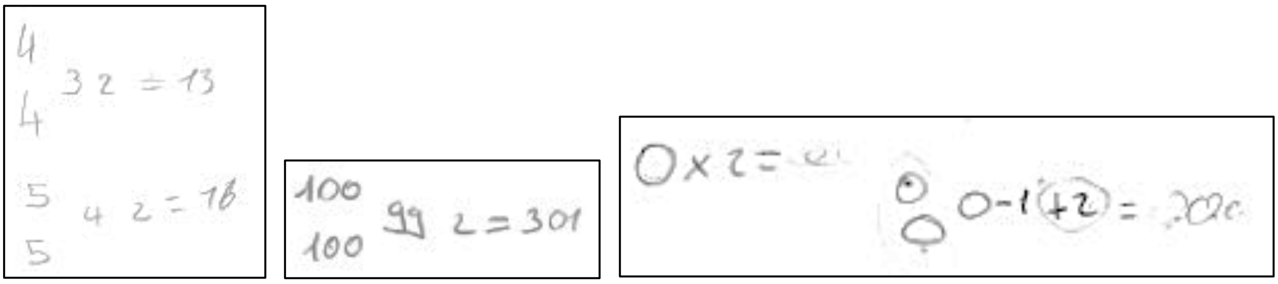

Figure 2: Ambre’s algebraic generalisations

The factual formula chosen by Ambre gives importance to the spatial arrangement of numbers (Figure 2, left and centre): the same number repeated twice in a column corresponds to the straws “at the top” and “at the bottom”, after that there is the number of straws in the “intersections” and finally there is always 2 that corresponds to the straws at the two extremities. The proposed formula encloses the spatial arrangement of the pattern and the student's temporal interpretation of it. The spatial arrangement of the formula does not include the operator signs, and the various numerical terms are simply juxtaposed, leaving the summation operation implicit.

When asked to find the number of straws of a non-particular number of squares, Ambre firstly generalises arithmetically the pattern. In the following excerpt, the student highlights the recursive relationship that links each stage and the next one: by adding a square, three straws are added.

Ambre: . . . if. . .we are told for example 2 squares of straws, we just have to add then 3 so that it makes 2 squares. . . Then it's kind of like this, you can do it over and over again. But once you know that with 1 square, there are 4, but with 2 squares, there are [*she counts the straws*]... There are 7. Well, we can repeat that every time.

The previous relationship is extendable to each stage of the pattern because, as Ambre says, “we can repeat that every time”. She identifies the regularity of the pattern and generalises it. In this generalisation, however, she does not offer a compact formula for immediately finding the number of straws needed for any number of squares. But, after some researcher’s interventions and questions, in addition to an arithmetic view of the pattern, Ambre also proposes an algebraic generalisation that has some characteristics of a contextual generalisation and some of the symbolic (Figure 2, right). This is evident in the following excerpt.

Researcher: How would you do, if you have any number of squares, to figure out how many straws?

. . .

Ambre: If we take the number in question... I don't know, I just represent the number with a circle.

Researcher: That's fine.

Ambre: We'll just have to make this circle in question time 2, equal to any number since it's a number we don't know [*she writes "Ox2=ooo", as shown in Figure 2, right*], but then, since there are all those that are on the bottom... Let's say that if there is… This number is the number of times that there is 1 square with a certain number of straws, that means... Anyway, we have to do, since it has this number of straws which will be 2 times in the calculation, it will be 2 times anyway [*she writes "O" and "O" in the column, as shown in Figure 2, right*], otherwise...

Researcher: Yes.

Ambre: And there will always be this number minus 1 in the rest of the calculation. Plus, if we always assume that it's squares, plus 2 afterwards [*she adds "O-1+2" at her formula, as shown in Figure 2, right*].

Researcher: Mh mh.

Ambre: Which will always solve to ta-ta-ta... I mean, these [*writing "=ooo", as shown in Figure 2, right*]…

Researcher: What is your ta-ta-ta?

Ambre: It's a number. A final number. But in itself, the aim of the calculation is to take 2 times the number we have...

Researcher: Yes...

Ambre: Minus that number... Minus 1 of that number, plus what we found there [*by circling the two "O O" in the column*], plus then the 2 that will come afterwards. And we're going to find the final number [*by indicating "ooo"*].

The generalisation proposed in the previous excerpt loses the reference to the particular number and can be implemented for any number. It straddles the contextual and the symbolic generalisation. The student in fact, after making the variable explicit by calling it "the number in question", represents it by introducing the symbol of a circle. It is a compact symbol here that has the same characteristics as algebraic alphanumeric symbolism, which the student has not yet encountered in her schooling. On this variable-circle, the student operates nimbly. At first, as we can see in Figure 2 on the right, she wants to multiply the variable by two. However, unable to complete the formula in such a decontextualised manner, the student abandons this attempt and returns to referring to the spatial structure of the pattern, the same structure she had already highlighted in her factual formula (Figure 2 left). So, although the variable is represented by a compact symbol, the generalisation proposed by the student still refers to the spatiotemporal context. In fact, she places the circle variable twice in a column, representing the number of straws at the top and the number of straws at the bottom of the pattern. Then she places the variable minus one for the straws at the "intersection" and she adds 2 for the two straws closing the pattern. The formula is in this aspect contextual, because it refers to the spatial arrangement of the pattern. In the student's words we clearly read these references, spatial such as "those on the bottom" or "here", and temporal such as "then" or "afterwards". Furthermore, the different parts of the formula are only partially joined together by operator symbols: when there is a number the operator is written ("-1" and "+2" in Figure 2, right) but variables-circles are not preceded by an operator, they are simply juxtaposed. The operators are instead verbalised by Ambre, who repeatedly expresses that the operation to be done is "plus".

Despite the contextual characteristics, the generalisation produced by the student fits any number and has the ambition to detach itself from the context.

Discussion

In the previous pages, we observed the use of algebraic thinking by a student diagnosed with dyscalculia and dyslexia. Despite the severe difficulties in mathematics identified by the diagnoses, the student was able to show, during the interview, that she was able to resort to algebraic thinking and practised various types of generalisations, including very fine ones. The student generalised arithmetically, but also algebraically: she developed generalisations with factual, contextual and standard symbolic characteristics. This shows the possible richness of the algebraic thinking of students with MLD, making an original contribution to the literature that focuses in most cases on what students cannot do rather than what they can do. This result is in line with research which focuses on other aspects of mathematical reasoning of pupils with MLD (i.e. proof, Gregorio, 2022).

Moreover, this case study offers an example of a pupil with definite difficulties in arithmetic who nevertheless shows a good command of the generalisation of geometric algebraic patterns. The identification of the structure of the situation and its generalisation were also preferred at the expense of mere counting or arithmetic generalisation. The discrepancy between algebraic and arithmetic skills shows that one is neither necessary nor sufficient for the other (neither arithmetic for algebra nor vice versa), constituting a further confirmation of the dividing line between the two disciplines (Radford, 2018; Kieran, 2004).

This result opens up access to the potential for developing algebraic thinking even for these students in great difficulty, for whom there are often minimal learning objectives. Indeed, solving problems similar to the one described in this article is not often offered to students with mathematical difficulties (Roiné & Barallobres, 2018). In general, for this type of student there is a preference for operative tasks where a lot of attention is given to the result and the product, to the detriment of the process and the structure of the mathematical situation.

This result also calls into question the types of tests currently used to identify MLD. The strong focus on basic arithmetic skills does not allow for the moment to differentiate between different types of difficulties in mathematics, identifying all and only those students with MLD as having arithmetic-calculator difficulties. There is a need to include other item types in the tests (Baccaglini-Frank et al., 2020) in order to offer a truly comprehensive screening of mathematical skills and difficulties, so as to cover the algebraic area, but also, for example, the spatial or geometric area.

In addition, the interview shown in the previous paragraphs highlights the fact that students diagnosed with the current tests have difficulties in some areas of mathematics, the ones tested for the diagnosis, and may show skills equal to their peers in other areas. This emphasises the importance of categorising students not only through their difficulties, but also through their competences, in order to offer them a learning framework best suited to them. This result is consistent with some recent research, which is interested in the creation of diagnostic tests to differentiate between different profiles of students with MLD (Baccaglini-Frank et al., 2020).

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