Journal of Mathematical Behavior 69 (2023) 101051

Contents lists available at ScienceDirect

Journal of Mathematical Behavior

journal homepage: www.elsevier.com/locate/jmathb

Teachers' mathematical problem-solving knowledge: In what way is it constructed during teachers' collaborative work?

Stéphane Clivaz [a,*], Valérie Batteau [a], Jean-Philippe Pellet [b], Luc-Olivier Bünzli [a], Audrey Daina [a], Sara Presutti [a]

[a] *UER MS and 3LS, Lausanne University of Teacher Education, Switzerland*
[b] *UER MI, Lausanne University of Teacher Education, Switzerland*

A R T I C L E I N F O

Keywords:
Teachers' mathematical knowledge
Problem-solving
Dialogic analysis
Lesson study

A B S T R A C T

This research aims to analyze the type of mathematics problem-solving knowledge for teaching used when working collaboratively in a Lesson Study (LS) process and examine how dialogic interactions contribute to knowledge construction. Five meetings during one LS cycle of a group of eight Swiss primary teachers were video recorded, transcribed and coded with the help of qualitative data analysis software. This analysis is conducted by crossing theoretical frameworks from two different fields in education, namely mathematics education and dialogic analysis. The mixed-method uses quantitative analysis with Markov chains and cross-tables, as well as qualitative analysis at micro-, meso- and macro-levels. This research suggests that participants collectively use their mathematics and their problem-solving content knowledge to focus on pedagogical problem-solving knowledge, that they navigate between different knowledge levels and that the roles of teachers and facilitators are differentiated but are also coequal.

1. Introduction

Mathematical knowledge necessary for teaching has attracted much attention in mathematics education (see, e.g., the report by Ball, 2017). The same tendency is exhibited in problem-solving, seen from the students' point of view and, more recently, the collective work of teachers accompanied by mathematics researchers and trainers (Borko & Potari, 2020). In our research, the teaching of problem-solving is studied from the point of view of the development of teachers' professional knowledge in a dialogical process within a lesson study (Lewis et al., 2019). With this objective in mind, we are currently analyzing the work of a lesson study group composed of eight teachers from grades 3 and 4 in the Lausanne region (French-speaking part of Switzerland) and two facilitators. From 2018 to 2019, this group carried out two lesson study cycles with the question: "how to teach grade 3–4 students to solve mathematical problems". Our research focuses on the first cycle.

The long process of creating our theoretical framework and the analytical grid was described in a recent article (Clivaz et al., 2023). We will take up certain concentrated elements of this and will refer to them in the theoretical and methodological section at the beginning of this article (Sections 2 and 3). The analysis and the discussion section will start with some results of the quantitative statistical analysis; this will be illustrated with our ongoing qualitative analysis and will lead to a temporary conclusion regarding the construction of mathematical problem-solving knowledge during teachers' collaborative work.

* Correspondence to: HEP Vaud, Cour 33, 1014 Lausanne, Switzerland.
E-mail address: stephane.clivaz@hepl.ch (S. Clivaz).

https://doi.org/10.1016/j.jmathb.2023.101051
Received 31 May 2022; Received in revised form 23 February 2023; Accepted 16 March 2023
Available online 30 March 2023

2. Theoretical framework

2.1. Lesson study

Jugyou Kenkyuu, literally "lesson study" (LS) came into being in Japan in the 1890 s. It was popularized in the 2000 s following international TIMSS comparisons (TIMSS Video, n.d.) and the comparison between mathematics education in Japan, Germany and the USA, presented by Stigler and Hiebert (1999) in *The Teaching Gap*. Thanks to the efforts made to promote LS, and in particular, the work of Lewis, who contributed to formalizing and popularizing LS in the USA (Lewis, 2002, 2015; Lewis & Hurd, 2011), LS was introduced in the USA as a professional development approach to improve US mathematical classroom teaching and learning (Yoshida, 2012). As a mode of professional development, LS has developed all over the world and has attracted the interest of many researchers in educational sciences, particularly in mathematics education (see, e. g., some of the most recent international edited books, Huang et al., 2019; Quaresma et al., 2018).

LS is often represented as a cyclic process with distinctive phases (Fig. 1). LS starts from an area of difficulty in teaching and learning, identified by a group of teachers. Teachers analyze the targeted learning, study the mathematical concept, consult the various teaching methods and study articles from professional journals and other resources. This phase, often named after the Japanese term, *kyozai kenkyu* (Watanabe et al., 2008), allows them to plan a lesson together. This lesson is implemented in the classroom of one of the group members. Other teachers observe the lesson in real-time and analyze its impact on students' learning. The group may decide to plan an improved version of the lesson to be conducted in another teacher's classroom and the loop begins again. The results of the work are disseminated, both in the form of a detailed lesson plan for future use by other teachers and in the form of articles in professional journals.

LS groups are usually led by an experienced teacher or trainer, known as a facilitator, who "keeps the conversation moving and fair. Involves all participants. Follows an agreed-upon agenda" (Lewis & Hurd, 2011, p. 124). While in Japan, LS is sometimes facilitated directly by the group members; it almost always involves one knowledgeable other, who provides feedback during the discussion after the research lesson and sometimes, another knowledgeable other, who can draw attention to key elements during the planning phase (Watanabe & Wang-Iverson, 2005). In countries where LS is developed (particularly in Japan), the role of facilitators as facilitators participating in the group and that of occasional external experts is very well defined. These two roles are often assumed by the same person or are combined where LS is starting to take root (Clivaz & Takahashi, 2018).

2.2. Mathematical Knowledge for Teaching Problem-Solving (MKTPS)

Most of the research regarding problem-solving has considered the student's point of view (for a survey of the state-of-the-art, see Liljedahl et al., 2016). A few authors have considered the teachers' point of view and the practice-based framework of the mathematical knowledge for teaching (Ball et al., 2008) to characterize the knowledge teachers use to teach problem-solving. In a somewhat parallel approach, Chapman (2015) distinguishes six categories of mathematics problem-solving knowledge for teaching based on a literature review from a sample of studies from 1922 to 2013 (Table 1).

Quoting Mayer and Wittrock (2006, p. 287), she considers problem-solving as "a form of cognitive processing you engage in when faced with a problem and do not have an obvious method of solution". She divides these types of knowledge into problem-solving content knowledge and pedagogical problem-solving knowledge (Fig. 2). All these categories are influenced by the teachers' problem-solving proficiency and by their affective factors and beliefs. We have made these conceptions of problem-solving and this categorization our own. Moreover, in line with Chapman's findings, for the purpose of bridging the categories of mathematics problem-solving knowledge for teaching and mathematical knowledge for teaching, we propose the following graphical representation of this categorization (Fig. 2).

We then needed to determine knowledge levels (Table 2) to identify the participants' knowledge evolution throughout the meetings

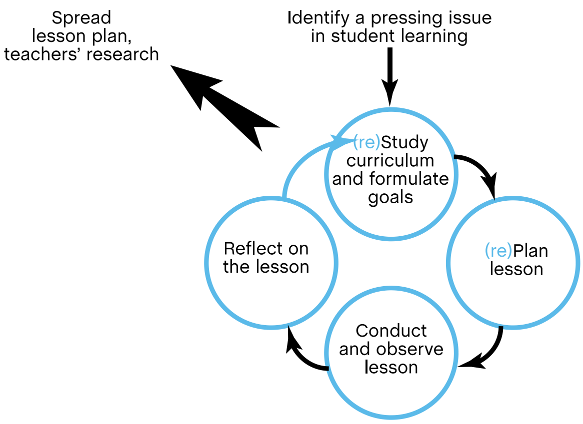

Fig. 1. LS cycle, adapted from Lewis et al., (2006, p. 4).

Table 1
Mathematics problem-solving knowledge for teaching Chapman (2015). The italicized parts are our contributions.

Knowledge of mathematical problems	Understanding of the nature of meaningful problems; structure and purpose of different types of problems; impact of problem characteristics on learners
Knowledge of mathematical Problem-Solving (PS) (*do/meta)*	Being proficient in PS *(do)*. Understanding of mathematical PS as a way of thinking; PS models and the meaning and use of heuristics; how to interpret students' unusual solutions; and implications of students' different approaches *(meta)*.
Knowledge of problem posing	Understanding of problem posing before, during and after PS.
Knowledge of students as mathematical problem solvers	Understanding what a student knows, can do, and is disposed to do (e.g., students' difficulties with PS; characteristics of good problem solvers; students' PS thinking).
Knowledge of instructional practices for PS	Understanding how and what it means to help students to become better problem solvers (e.g., instructional techniques for heuristics/strategies, metacognition, use of technology, and assessment of students' PS progress; when and how to intervene during students' PS).
Knowledge of affective factors and beliefs (*teacher/student)*	Understanding nature and impact of productive and unproductive affective factors and beliefs *(of the teachers/of the students)* on learning and teaching PS and teaching.

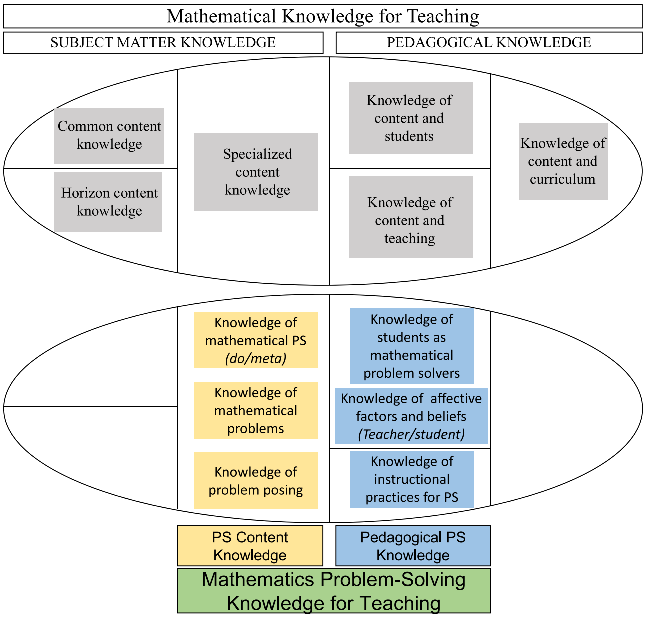

Fig. 2. Mathematical Knowledge for Teaching Problem-Solving (MKTPS). Mathematical knowledge for teaching (upper grey part of the figure) is taken from Ball et al. (2008); mathematics problem-solving knowledge for teaching (colored categories) is from Chapman (2015); graphical representation of the colored categories is by the authors of this paper (Clivaz et al., 2023).

or differences related to the roles (teacher, facilitator). These levels do not constitute a hierarchy: contextualized knowledge or questioning is as valuable as generalized knowledge. The knowledge levels are labeled from 1 to 5.

After having presented the types of knowledge, we will study the different ways in which it can be expressed in the course of the dialogue. More precisely, we will focus on characterizing the dynamics of the interactions and describing our theoretical framework and our analysis approach.

2.3. Interaction analysis

Comprehensive research on LS groups and the fact that they appear to have an impact on teachers' professional knowledge often focuses on the essential role of facilitators (e.g., Bjuland & Helgevold, 2018; Lewis & Hurd, 2011; Lewis, 2016) and, potentially, knowledgeable others (e.g., Seino & Foster, 2020; Takahashi, 2014). While many studies mention the importance of these roles and give examples of facilitator interventions or mention statements by teachers regarding the importance of this role, to date, qualitative studies describing precisely how this role allows teachers to build professional knowledge are rare.

In our previous research, we examined the evolution of the trainer's role, in terms of knowledge sharing, in a series of LS (Clivaz & Clerc-Georgy, 2020) and showed which mathematical knowledge for teaching is used by teachers during the LS process (Clivaz & Ni Shuilleabhain, 2019; Ni Shuilleabhain & Clivaz, 2017). Nevertheless, interactions within the group, particularly between the facilitators and the teachers, are yet to be explored.

With the objective of accurately describing how teachers' knowledge is constructed or evolves and, more generally, to better

Table 2
Knowledge levels (Clivaz et al., 2023).

Knowledge levels	
1	Inaccurate knowledge, lack of knowledge, self-assumed ignorance, and/or debatable personal representation
2	Unexplained knowledge. Observation, testimony
3	Incomplete knowledge, knowledge with a low degree of certainty. Explicit questioning
4	Contextualized explicit knowledge. Speaker knows or appears to know the rationale
5	Generalized, decontextualized knowledge (decontextualization process, possibly not fully completed). Generic example

understand what happens between actors within an LS process, we have been led to focus on discourse analysis from a sociocultural perspective. This perspective is rooted in the work of Vygotsky (1962, 1978), for whom the acquisition and use of language transform children's thinking. One of our first inspirations was driven by the work of Vermunt and his colleagues (Vermunt et al., 2019; Vrikki et al., 2017; Warwick et al., 2016), who categorized the dialogic processes in LS groups, in order to find statistical correlations between certain dialogic features and teachers' meaning-oriented learning in LS. Since these categories were too broad for a comprehensive analysis, we were led to study the work of a sister group within the Cambridge Educational Dialogue Research group (CEDiR, n.d.), namely, the Scheme for Educational Dialogue Analysis (SEDA, n.d.; Hennessy et al., 2016) group.

2.3.1. *The construction of the LS interaction analysis grid*

The construction of our Lesson Study Dialogue Analysis (LSDA), based on the Scheme for Educational Dialogue Analysis grid (Hennessy et al., 2016; SEDA, n.d.), is described in detail in our team's theoretical and methodological paper (Clivaz et al., 2023). The result is a three-level analysis grid. These three levels of analysis are nested and provide a systematic structure for the analysis of the data:

- *Communicative acts* are at micro-level. They correspond to an utterance or turn of speech,[1] produced by one person. They are identified by their function in the interaction (asking a question, justifying, etc.), using one of the codes relating to clusters E, Q, R, P and G, as described in Table 3. The transitions between two communicative acts are also at micro-level.
- *Communicative events* are at meso-level. Our objective is to highlight the connections made in the course of exchanges. Therefore, this coding level is carried out by blocks of several turns of speech, making it possible to highlight a form of sequencing of the interactions. A block corresponds to a sequence of interactions connected to the same reference: previous contributions, the research lesson, a teaching experience, a personal experience, a teaching representation, a reference, the LS process (code C in Table 3).
- *Communicative situations* are at macro-level and represent the context in which the conversation takes place, linked to the phases of the LS process: choice of topic, study of the topic, planning of the lesson, research lesson, analysis of the lesson.

3. Data and method

3.1. *Context, data collection and data coding*

The data analyzed in this paper are part of the work of a LS group composed of eight grade 3 and grade 4 teachers from the Lausanne region and two facilitators. The two facilitators are a mathematics educator (the first author of this contribution) and a teacher from the institution, who has participated as a member of a previous LS group in mathematics. The group's initial plan was to focus on teaching how to solve problems. During the first meeting, they discussed the difficulties as regards the teaching and learning of problem-solving and decided to focus on helping the students to have a representation of the problem. During the second meeting, they discussed eight different problems successively. From a professional article focusing on problem-solving, they distinguished between different kinds of problems to help them in their choice. During the third meeting, they decided to use a problem from an external state test, with which their students had struggled two years previously (see Appendix). This problem was used in the research lesson. During the fourth meeting, the group discussed the representation of the problem, how the students would solve it and the mathematics at stake. Then, the group members planned the research lesson. During the post-lesson meeting, they successively discussed their observations of students' groups. They also discussed the importance of the manipulatives, the whole-class discussions, the mathematical aspects and the way in which to help struggling students.

We focus here on the first five meetings of this group. The main topic of each meeting and its duration are shown in Table 4.

Part of the meeting in which the conversation was not related to the LS theme (discussion off subject, organizational topics…) was not transcribed. The five meetings represent 1875 transcribed utterances, 1161 are coded with one unique LSDA code each. These contain 1439 MKTPS codes, each associated with one unique knowledge level.

The data were transcribed and coded, using a qualitative data analysis program, Transana (transana.com), by a team of four of the authors of this paper, over a two and a half-year period. This software allows us to transcribe the dialogue of the five meetings and to

[1] Our data do not contain long conversation, and we considered it unnecessary to distinguish between utterance and turn of speech. Therefore, these are equivalent in our data, and we are using both terms.

Table 3
Codes for LS Dialogue Analysis (Clivaz et al., 2023).

LSDA clusters	Features and LSDA codes
E – Express or invite new ideas	This category marks the entry of a new subject into the discussion, a new idea, an observation. Distinction between invitations to formulate new ideas (EI) and expression of a new idea (EE) is made.
Q – Arouse development or reasoning	This category is used with the next category R to code a series of exchanges around a subject. The Q-coded turn involves reference to a previous input. The three possible purposes of the Q-coded turn are, to better understand a factual statement (QC) or to understand the reasons for a previous statement (QJ) or to consider other possibilities or hypotheses (QH).
R – Answer, develop	This category has three possible purposes: to provide clarification and explanation (RC), to give a justification (RJ), to develop other possibilities or hypotheses (RH).
P – Position or coordinate	This category is used to indicate a turn intended to mark one's stance or to coordinate ideas in relation to previous exchanges. It may involve synthesizing ideas (PS), evaluating different perspectives (PE), challenging an idea (PV) or taking a position (PP), approving (PA).
G – Guide	This category is used to indicate a turn intended to guide the course of interaction by encouraging dialogue (GG), by verbalizing the rules of communication in order to promote discourse (GD), by proposing an immediate action (GA), by proposing an action in the future (GP), by taking an expert position (GE), by focusing (GF).
C – Connect	This category is used to show what a series of exchanges refers to. It might refer to: the content of a past discussion episode (CA) the research lesson (past or future, CL) one's teaching experience (CE) one's personal experience (CH) the LS process (at a meta level, CS) believes about teaching and learning (CB) the learning trajectory of participants (CT) the mathematical task (CM)

Table 4
The five meetings analyzed.

Meeting number	Main topic	Total duration [minutes]	Transcribed part [minutes]	Number of transcribed occurrences
1	General discussion about problem-solving teaching	71	36	103
2	Study of the curriculum material	86	70	350
3	Study of the curriculum material, choice of problem for the research lesson	92	89	557
4	Planning of the research lesson	90	51	311
5.1	Research lesson, taught by T7,observed by the group	43		
5.2	Post-lesson discussion	94	82	554

code the interactions according to the analysis grids we have just presented. All the transcripts and the analysis are written in French. An English translation is only provided for papers in English.

The ability of the software to code the transcript, linked to the video, is essential during the coding phase, which sometimes involves taking into account the tone of voice and the interaction's non-verbal context to decide on the code. Once the coding is complete, it is possible to carry out the first part of the analyses using Transana, making it possible to cross-reference and link these codes or create tables and diagrams that give an overall view of the coded data. This also highlights elements of descriptive analysis: speaking time per speaker, types of knowledge and levels and LSDA clusters. Data can then be exported for further processing in other tools.

3.2. Method

For the research, we adopted a mixed method of analysis: (a) qualitative coding of utterances; (b) quantitative analysis with Markov chains modeling transitions of knowledge, levels of knowledge, and interactions, in addition to cross-tables to analyze momentary co-occurrences and (c) qualitative analysis at micro-, meso- and macro-levels. The main question of this research deals with the construction of mathematical knowledge for teaching related to problem-solving, during the LS process. It involves taking into account the temporality in our statistical model. We chose the formalism of *Markov chains* (Gagniuc, 2017) to explore temporal dependence; Markov chains are easily visualizable models describing, for a given time-varying discrete variable of interest, its probability of transitioning to a given value at the next time step, depending on its value at the current time step. We analyzed the sequences and transitions of knowledge during the LS meetings, in order to grasp the construction of knowledge. To do so, the data were exported from Transana and we generated the Markov chains of interest with our own software. The variables, the transitions of which we analyzed with Markov chains, are: MKTPS, the knowledge levels and the interactions (LSDA cluster and LSDA code).

We exemplified the definition of Markov chains with the MKTPS variables (mathematical knowledge for teaching related to problem-solving).

$(X_n)_{n\geq 0}$ was the variable MKTPS taken at the time *n*. $(X_n)_{n\geq 0}$ has qualitative values $(x_i)_{i\geq 0}$: knowledge of students as mathematical problem solvers, knowledge of instructional practices for problem-solving (PS), knowledge of the affective factors and beliefs of teachers and students, knowledge of problem posing, knowledge of mathematical PS (do or meta), knowledge of mathematical problems, common content knowledge, horizon content knowledge, specialized content knowledge, knowledge of content and students, knowledge of content and teaching, knowledge of content and curriculum.

$$\forall i \in \mathbb{N}, P(X_{i+1} = x_{i+1}/X_i = x_i) = P(X_{i+1} = x_{i+1}/X_i = x_i, \ldots, X_0 = x_0)$$

We make the hypothesis that the probability of a MKTPS appearing at the time $i+1$ only depends on the MKTPS at the time i and not before (0 until $i-1$). This embodies the so-called *memoryless* property needed for Markov chains: the sequence of values leading to time i does not matter. We then calculate the probabilities of transition from X_i to X_{i+1} $\forall i \in \mathbb{N}$. We do so for each utterance of each LS meeting, and then for all LS meetings taken together to obtain a global view.

Should there be a type of knowledge coded at the time i and no knowledge coded at the time $i+1$, we decided to skip time $i+1$ when computing the transition probability and we considered the transition between time i and time $i+2$ instead (or i to $i+3$ or i to $i+4$, etc. until we found an utterance with knowledge). We checked and confirmed this choice with a qualitative view. The rationale is for utterance coding with interactions (LSDA) and without knowledge; the discussion and the evolution of knowledge continue after these utterances without knowledge.

We are dealing with a methodological issue: some utterances have been coded with several kinds of knowledge and each kind of knowledge can be associated with a different knowledge level. This contradicts our setting that each time step can be affected by a single value of the variable of interest. In this case, we chose to normalize as follows: for example, (Fig. 3), six utterances (out of 86 utterances) are coding knowledge of mathematical PS during meeting 1. One utterance after these six utterances is coded with four types of knowledge: knowledge of affective factors and beliefs (students), knowledge of students as mathematical problem solvers, knowledge of problem posing and knowledge of instructional practices for PS.

This means:

X_{i+1} = k. of affective factors and beliefs (students), k. of students as math. problem solvers, k. of problem posing, k. of instructional practices for PS, k. of math. PS (meta).

and X_i = k. of math. PS (meta).

Then, during the computation of the total number of transitions from any state to any other, used to form the resulting transition

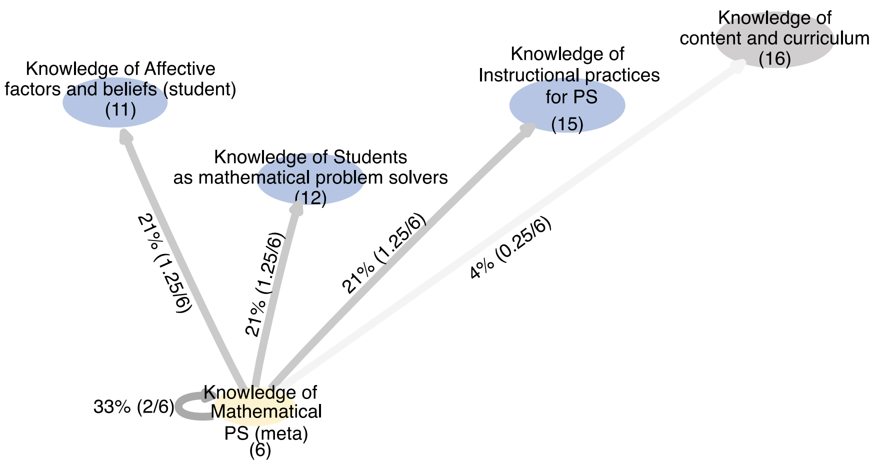

Fig. 3. Markov chains for MKTPS in meeting 1.

probabilities, this particular transition was taken into account as a contribution of one fourth of a transition from the state at time *i* to each of the four states found at time $i+1$. This explains the non-integer transition counts, labeling certain transition arrows in Fig. 3.

$$P(X_{i+1} = \text{k. of affective factors and beliefs (students)}/X_i = \text{k. of math. PS (meta)}) = \frac{1,25}{6} \approx 21\%$$

$$P(X_{i+1} = \text{k. of students as math. problem solvers}/X_i = \text{k. of math. PS (meta)}) = \frac{1,25}{6} \approx 21\%$$

$$P(X_{i+1} = \text{k. of instructional practices for PS}/X_i = \text{k. of math. PS (meta)}) = \frac{1,25}{6} \approx 21\%$$

$$P(X_{i+1} = \text{k. of content and curriculum}/X_i = \text{k. of math. PS (meta)}) = \frac{0.25}{6} \approx 4\%$$

$$P(X_{i+1} = \text{k. of math. PS (meta)}/X_i = \text{k. of math. PS (meta)}) = \frac{2}{6} \approx 33\%$$

A Markov chain, such as that depicted in Fig. 3, should be read as follows. Each vertex represents a possible value for the variable of interest *X* (here, MKTPS). Vertices are also labeled with an integer representing the number of times *X* took this value. The long diameter of the vertices is proportional to the log of that number (the log chosen, so as to provide a decent output for both small and large values). The colors of the vertices are linked to the values described in Section 2. Outgoing arrows represent transitions found in the data from one value to another. Transition edges are labeled both with a (rounded) percentage denoting transition probabilities and with absolute numbers, allowing to add some perspective to the percentages. Edges are drawn with darker shades of gray as the percentage gets larger. Rare transitions make the Markov chain diagrams difficult to read, since the resulting graph can become very dense. Therefore, from now on, we choose to represent and consider only those transitions the probability of which is higher than a certain value. After conducting tests, it appeared that an adequate threshold was 10%. Thus, from now on, for example, the arrow corresponding to 4% in Fig. 3 will no longer be visible in our diagrams.

In this paper, we present the analysis of Markov chains for the MKTPS. We perform the same analysis with knowledge levels, with LSDA (Table 3) and at a different analysis grain with the LSDA codes (Table 3).

In addition to these graphs of Markov chains, we realize cross-tabulations of the variables two by two: MKTPS, knowledge levels, speakers and LSDA (or LSDA cluster). We complete this analysis with time charts crossing the utterer, the MKTPS and the knowledge level per LS meeting.

We adopt a mixed methodology in two ways. The quantitative findings are crossed with qualitative analysis, by considering illustrations at the meso-level of *communicative events*: several utterances. The quantitative findings are also illustrated with qualitative analysis at micro-, meso- and macro-level. We use the succession of two utterances (micro-level), the content of the sequence of interactions (meso-level) or the context of each phase of the LS process (macro-level) to explicate the quantitative findings.

3.3. Research questions

In the case of our study, we needed to combine MKTPS and the LSDA frameworks to "get a multi-faceted insight into the empirical phenomenon in view" (Bikner-Ahsbahs & Prediger, 2010, p. 496). We refer to "combining" these frameworks in the sense of Prediger et al. (2008). The conceptual framework captures two different points of view on teacher dialogue during a LS, as reflected in our research questions, heuristically structured around the main topics: PS in mathematics and the LS process.

Our main research question, "In what way is MKTPS built collectively during the LS process?" is addressed through the analysis of the Markov chains for the MKTPS. This collective construction of MKTPS at micro-level is analyzed by studying the transitions of MKTPS between two utterances. We chose to focus mainly on the collective dimension in our study of MKPTS. By comparing these quantitative results with the content of communicative situations, communicative events and communicative acts, our mixed-method allows us to answer the following sub-questions: "What is the MKTPS that emerges during each LS meeting?"; "What are the transitions of MKTPS and those most represented in each LS meeting and in all meetings?" and "How do types of MKTPS follow each other in the dialogue?".

Furthermore, we analyze the Markov chains for the knowledge level and cross findings with *communicative situations* at macro-level. These analyses lead to research question 2: "Which knowledge levels are used during each LS meeting? Do they evolve?" This question is more individualized in order to take into account the potential particularities of the facilitators: "What are the characteristics of the facilitators in terms of the levels of knowledge expressed?".

In addition to the exploration of value transitions, as can be depicted through Markov chains, we subsequently show tables crossing two of our variables of interest (e.g., Table 9 below), demonstrating the percentage of utterances where two given variable values coincided. We refer to these as cross-tables later in the text. The analyses of the Markov chains for the LSDA codes, for LSDA clusters and the cross-tables of the LSDA cross utterers allow us to deal with the following research questions: "What are the dynamics of the interactions of LS dialogue analysis at the micro-level of communicative acts?" and the particularization of the facilitators and the teachers: "Which types of interactions during LS meetings characterized the facilitators compared to those used by the teachers?".

The research questions and sub-questions, the level of activity (see 2.3.1) and the method to answer the questions are summarized in Table 5).

The next section highlights the analysis based on the presented method and the main findings that answer the research questions and sub-questions.

4. Findings

4.1. Mathematical knowledge for teaching related to problem-solving (MKTPS)

4.1.1. MKTPS for all meetings and per meeting

During the five LS meetings analyzed, 1439 MKTPS codes were assigned. For the globality of the five LS meetings (Table 6), the most prevalent MKTPS was knowledge of instructional practices for PS (423 out of 1439 MKTPS codes) and knowledge of students as mathematical problem solvers (360 out of 1439 MKTPS codes).

The high number of these two categories is interpreted per meeting at the macro-level of communicative situations. Meeting 1 begins with a round-table discussion on PS teaching and teachers' representations of teaching and PS. The main MKTPS in this meeting is knowledge of content and curriculum (16), followed by knowledge of instructional practices for PS (12) and knowledge of students as mathematical problem solvers (15). This can be interpreted by the large number of references teachers make to tasks from textbooks, when talking about their teaching of PS (knowledge of content and curriculum).

Meeting 2 and meeting 3 deal with the study of the curriculum material. The most represented MKTPS is knowledge of instructional practices for PS (48 during meeting 2 and 151 during meeting 3). As exemplified in the following extract (Table 7), utterers speak about the syllabus, in particular, mathematics teaching in kindergartens and the development of social competencies. They mainly use knowledge of instructional practices for PS and the consequences on student learning (T7 at 50:29 and 51:46).

During the next meeting, involving the planning of the research lesson, the most represented MKTPS are knowledge of instructional practices for PS (98 utterances) and knowledge of students as mathematical problem solvers (72 utterances). This illustrates the fact that, in order to plan a lesson, teachers and facilitators use almost as much knowledge of students as they use knowledge of instructional practices.

The post-lesson discussion (meeting 5) focuses on students' learning during the lesson, but also on the teaching and the consequences of teacher choices on students' learning. Thus, the type of MKTPS mentioned most frequently is knowledge of students as mathematical problem solvers (151 utterances).

In summary, from one meeting to the other, we observed variations in the importance of the two teaching/learning MKTPS: the most represented knowledge is knowledge of instructional practices for PS during the study of the curriculum; both types of knowledge are equally represented during the planning of the lesson; knowledge of students as mathematical problem solvers is prevalent during the post-lesson meeting. The macro-level of communicative situations allows us to match these variations of the main MKTPS with the objectives of each phase of the LS. The difference in importance of the various types of knowledge and, for example, the focus on pedagogical types of mathematical knowledge for teaching echoes similar findings regarding mathematical knowledge for teaching during the numeration-related LS by Clivaz and Ni Shuilleabhain (2019).

The other numerous types of MKTPS involved in this LS process are knowledge of problem posing, knowledge of mathematical problems and knowledge of mathematical PS (meta) (between 120 and 170 utterances). The most prevalent mathematical knowledge for teaching is knowledge of content and curriculum (83 utterances). This can be explained by the fact that this knowledge is close to pedagogical knowledge related to PS. As a matter of fact, the discussions during the five meetings involved limited mathematical knowledge for teaching. One of the rationales is that the choice of the mathematical problem for the research lesson involves mathematical knowledge, which the students would regard as elementary. Mathematical knowledge is used as a tool inside the problem and not as an object of teaching. Furthermore, in utterances related to PS, mathematical knowledge for teaching is often

Table 5
Research questions and methods.

Research questions and sub-questions	Method	Level of activity
1. In what way is MKTPS built collectively during the LS process? • What is the MKTPS that emerges during each LS meeting? • What are the transitions of MKTPS and those most represented in each LS meeting and in all meetings? • How does one MKTPS follow on to the next in the dialogue?	Comparison of quantitative results (quantitative analysis of the Markov chains for the MKTPS) with the content of communicative situations, communicative events and communicative acts	micro meso macro
2. Which knowledge levels are used during each LS meeting? Do they evolve? In particular, what are the characteristics of the facilitators in terms of the levels of knowledge expressed?	Markov chains for the knowledge level, crossed with *communicative situations*	Link micro-meso-macro
3. What are the dynamics of the interactions of LS dialogue analysis at the micro-level of communicative acts? In particular, which types of interactions during LS meetings characterized the facilitators compared to those of the teachers?	Markov chains for the LSDA codes, for LSDA clusters. Cross-tabulations of LSDA cross speakers	micro

Table 6
Number of types of knowledge in each meeting.

	Meeting 1	Meeting 2	Meeting 3	Meeting 4	Meeting 5	All meetings
Knowledge of mathematical problems	2	52	83	6	7	150
Knowledge of mathematical PS (*do*)	1	18	11	3	7	40
Knowledge of mathematical PS (*meta*)	6	10	51	15	39	121
Knowledge of problem posing	2	16	52	58	35	163
Knowledge of students as mathematical problem solvers	12	42	83	72	151	360
Knowledge of instructional practices for PS	15	48	151	98	111	423
Knowledge of affective factors and beliefs *teacher*	1	8	1	1	0	11
Knowledge of affective factors and beliefs *student*	11	4	4	2	21	42
Common content knowledge	0	4	5	0	0	9
Specialized content knowledge	6	0	2	13	0	21
Knowledge of content and students	1	3	0	0	0	4
Knowledge of content and teaching	10	1	0	0	1	12
Knowledge of content and curriculum	16	28	37	1	1	83
Total	83	234	480	269	373	1439

Table 7
Extract of meeting 3 (50:29–51:46).

Time	Utterer	Transcript	LSDA	MKTPS	Level
0:50:29	T7	I agree with what T4 said, it requires a lot of skills other than mathematics, I mean, it requires being able to discuss this with your classmate, without having to talk about something else, without having to explain to your classmate “well yes, you do three plus two equals four”, instead you show him/her how…	EE	Common content K. K. of students as math. problem solvers K. of instructional practices for PS	1 4 4
0:50:51	T4	That’s five.	PP	Common content K.	2
0:50:52	T7	I’m sorry, five. And it’s true that this, I find that it’s also throughout a whole schooling that we work on it, and I think that they are … you know in grade 5 or 6, well, yeah, if we start in kindergarten, they have. they can’t do that.	EE	K. of content and curriculum K. of instructional practices for PS	4 4
0:51:18	T4	No, but you’d be surprised, I think, that in kindergarten, I mean, working in groups, listening, listening to others, that’s already worked out. in pre-K2, you know.	PP	K. of instructional practices for PS	4
0:51:36	T6	And then with the calculations, there’s already some.We associate two quantities, and we count the tokens, I think there’s a lot of things that are done too and I realize…	RJ	K. of instructional practices for PS	4
0:51:46	T7	No, but I mean exchange. The sharing between children, for the rest, I agree with T4, pupils can do that!	RJ	K. of students as math. problem solvers	4

implicit and “covered” by the mathematics problem-solving knowledge for teaching. Other examples of MKTPS are less well represented (less than 40 utterances).

4.1.2. Markov chains for the MKTPS

The Markov chain in Fig. 4 allows us to analyze the sequencing of two utterances coded by MKTPS for all LS meetings. One finding is that the probability of a type of MKTPS being the same at time $i+1$ as at time i is around 50% for several MKTPS (knowledge of students as mathematical problem solvers, knowledge of instructional practices for PS, knowledge of content and curriculum, knowledge of mathematical problems, knowledge of mathematical PS and common content knowledge). This is a consequence of our methodological choice of the turn of speech as a unit of analysis: the same topic is very often continued by the next utterer, and the same type of knowledge is therefore often present in the next utterance. For this reason, we focus our analysis on the changes of MKTPS that are not influenced by this choice of unit of analysis, namely, on the transitions between different MKTPS.

This Markov chain points out a hub consisting of the two most prevalent MKTPS: knowledge of students as mathematical problem solvers and knowledge of instructional practices for PS. This hub is attractive in that sense: all the visible arrows come from MKTPS outside of this hub. This observation should be interpreted as follows: after an utterance coding regarding any MKTPS, the probability that the next utterance will be coded by a type of MKTPS from this hub is high. For all LS meetings, almost all transitions between two utterances deal with knowledge of this hub or come down to this hub.

This confirmed that the LS process involved participants using their knowledge to refocus and to apply it to the knowledge of students as mathematical problem solvers and the knowledge of instructional practices for PS. We illustrate this phenomenon with the knowledge on problem posing for all meetings. The utterances coded with knowledge on problem posing are associated with utterance coding of the same knowledge (41%, see Fig. 4), e.g., utterances with knowledge of students as mathematical problem solvers (21%) and with knowledge of instructional practices for PS (21%). When a speaker talks about problem posing (Table 8), there is a high probability of the next speaker talking about the effect of the choice of problem posing on instructional practices or on students. In the following extract (Table 8), teachers and facilitators deal with the difficulties related to the contexts of problems and the vocabulary used when addressing problems. T1 and T6 speak about the changes in the wording of a problem (changing “videotape” to “CD”),

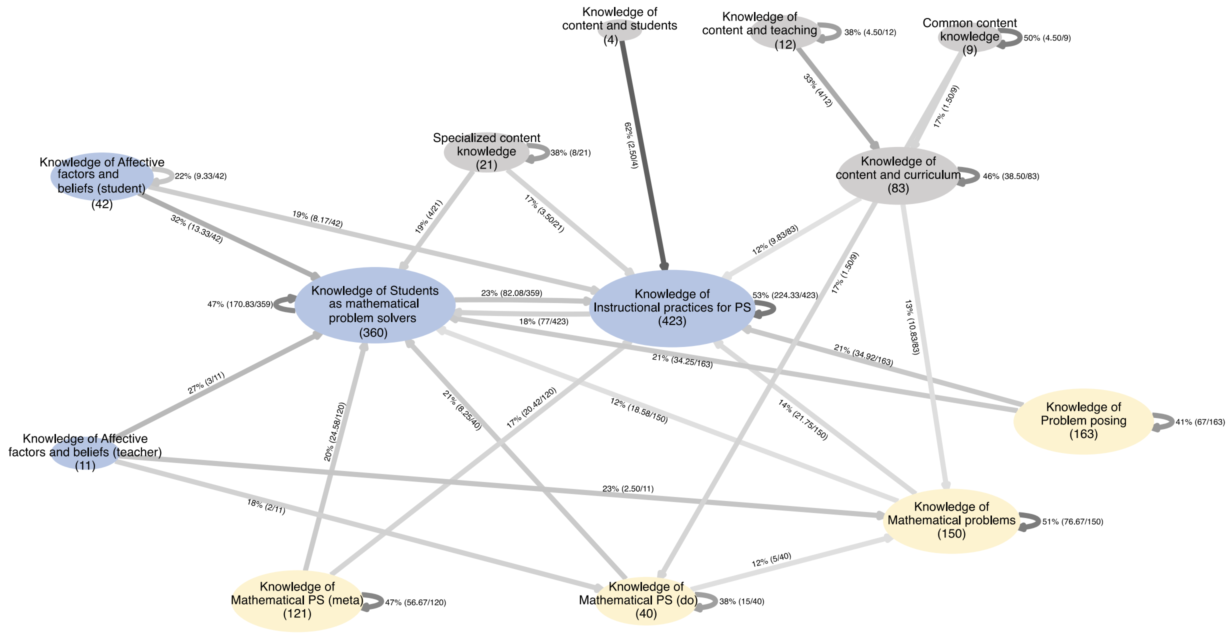

Fig. 4. Markov chains for MKTPS for the five LS meetings.

Table 8
Extract of meeting 3 (1:19:32–1:20:57).

Time	Utterer	Transcript	LSDA	MKTPS	Level
1:19:32	T1	You put "CD", or you put.	RJ	K. of problem posing	2
1:19:36	T6	No I don't put "CD" but, for me the objective is really to estimate a duration, and to read a table and then as a result.but it's true that I take away the cover story of the problem.	PP	K. of instructional practices for PS K. of problem posing	4 4
1:19:45	F2	But do we have to, we.			
1:19:48	F1	But it's two different objectives, it might be interesting, and that's what might be difficult here, […] I think we are really discriminating socially when we discuss this kind of problem [.]	EE	K. of problem posing	5
1:20:04	F1	I think there's a concern [about the problem dealing with compost and tree planting], I mean, we're first discriminating against the parents' job. For once it's more like the children of farmers who are advantaged, that's good! I mean there we're already separating those who know the context, and those who don't. I think that.But, in our case, do we want to work more on this question of.yeah, the context, of.maybe solving the problem without knowing the words, and then afterwards there are the difficulties of mathematical solving, or do we work more on mathematical solving. or both at the same time…	EE	K. of instructional practices for PS K. of problem posing	3 4
1:20:35	T6	It's going to be hard, right?	PA	–	
1:20:36	F2	But I… Do we need to clarify certain starting elements, well, so that, precisely so that they can start. Talking about the vocabulary essentially, but also about the context, without talking about the problem itself, what do we need to clarify so that they can understand?	EIG	K. of instructional practices for PS	3

necessary in order to make the vocabulary accessible to the students (knowledge of problem posing).

Likewise, when the speakers study the curriculum or plan the lesson, they use knowledge on problem posing and then talk about the effects of their choices on the instructional practices for PS or students' learning.

This section presents transitions between different MKTPS outside the hub, that do not go towards the hub. Knowledge of mathematical problems appears after knowledge of affective factors and beliefs (23%), knowledge of mathematical PS (do) (12%) and knowledge of content and curriculum (13%). The rationale is that knowledge of mathematical problems highlights two different aspects: one focused on the problem (nature, structure, purpose) and one focused on the impact of problem characteristics on learners.

Two other MKTPS, namely, knowledge of mathematical PS (do) and knowledge of content and curriculum are more than 10% likely to appear after another MKTPS. This observation reinforces the phenomena of the hub: there are few transitions (more than 10%) between two MKTPS outside the hub.

This section allows answering the sub-questions: "what are the transitions of MKTPS and those most represented in each LS meeting and in all meetings?" and "how do types of MKTPS follow one other in the dialogue?". This section shows dynamics in the transitions between MKTPS, with an attractive hub composed of the two main MKTPS.

The utterances coded with MKTPS only inform of the presence or absence of MKTPS. The next section completes this analysis with

Markov chains for knowledge levels, non-hierarchically classified from 1 to 5 and associated with each MKTPS.

4.2. Knowledge levels

4.2.1. Knowledge levels for all LS meetings and characteristics of facilitators

This part deals with research question 2: "Which knowledge levels are used during each LS meeting? Do they evolve? In particular, what are the characteristics of the facilitators in terms of the levels of knowledge expressed?" The most prevalent knowledge levels are level 2 (unexplained knowledge; observation, testimony), level 4 (contextualized explicit knowledge; speaker knows or appears to know the rationale) and level 3 (incomplete knowledge or a low degree of certainty; explicit questioning). These findings will be refined and explained by the phase of the LS process at macro-level.

Another observation is that level 5 (generalized, decontextualized knowledge and process or generic example) is sparsely represented (65 utterances). This limited appearance of level 5 can be explained by the posture of the two facilitators, who prepare the LS meetings together. Their profiles and postures are different: facilitator 1 (F1) is a mathematics education researcher (first author of this paper). At times during the meetings, he plays the role of a knowledgeable other (Seino & Foster, 2020). Facilitator 2 (F2) is an experienced teacher in the same school as the other teachers. She participated in a previous LS for two years as a teacher. F1 has 78% of knowledge level 5 and F2, 9%. The remainder (13% of utterances at coding level 5) is shared among the other teachers. The low level 5 should also be explained by the choice of a broader subject, i.e., PS compared to another circumscribed subject. Another finding is that 30% of the utterances of F1 are coded at level 4. Thus, the profile of F1 with regard to knowledge levels 4 and 5 is different from that of F2 and the other teachers. However, regarding knowledge levels 1, 2 and 3, the profile of F1 is similar to that of the other participants.

Codings at level 1 that concern inaccurate knowledge, lack of knowledge, self-assumed ignorance and/or debatable personal representation are sparsely represented (44 codes). This level is shared among the participants (Table 9).

A profile of the facilitators emerges: F1′s speech is mainly coded as knowledge levels 4, 2 and 5, F2 shares a similar profile with the other teachers, with most knowledge levels coded as 2, 3 and 4 (Table 10).

4.2.2. Markov chains for the knowledge levels

This section deals with research question 2 "Which knowledge levels are used during each LS meeting? Do they evolve?" The Markov chains for knowledge levels (Fig. 5) show that the probability of staying at the same level is relatively high for each level (between 23% and 58%) across all LS meetings. This is a consequence of our methodological choice of unit of analysis as shown previously for the Markov chain for the MKTPS.

After displaying inaccurate knowledge, a lack of knowledge, a self-assumed ignorance and/or a debatable personal representation (level 1), the probability of changing levels and moving to another knowledge level is high (77%). The probability of moving to knowledge level 3 (incomplete knowledge or knowledge with a low degree of certainty or explicit questioning) is 27%, to knowledge level 2 (unexplained knowledge or an observation or testimony) 22% and to knowledge level 4 (contextualized explicit knowledge) 22%. In that sense, the LS process involves speakers improving their knowledge level from a micro and collective point of view: the Markov chains take into account the transition of two utterances and each utterance corresponds to a speaker.

As an illustration of the transition between the levels 1, 2 and 3, in the following extract, teachers speak about the external state test and students' difficulty in using their knowledge to resolve these problems. T6 has a debatable personal representation of students: they do not remember what they learned. His knowledge of students as mathematical problem solvers is coded 1 for the knowledge level. Then, T4 repeats the words and transposes these to the knowledge of instructional practices for PS at level 2 (unexplained knowledge, observation, testimony).

These Markov chains show important transitions between the main knowledge levels 2, 3 and 4 for the globality of the five meetings (Fig. 5, all five meetings). The same analysis per LS meeting (rest of Fig. 5) shows particularities that can be explained at the macro-level of the communicative situation.

Meeting 1 mainly contains level 3 knowledge, i.e., the level of doubt and questioning or incomplete knowledge. The objectives of

Table 9
Cross-table, relative knowledge level cross utterers (for all LS meetings). Percentages are calculated column-wise in order to analyze who among the participants uses the most a given level of knowledge. Cells are colored in shades of red when they contain a value above the median and in shades of blue for values below this (white is neutral).

Knowledge level / Utterer	1	2	3	4	5
F1	0.04	0.13	0.15	0.30	0.78
F2	0.13	0.11	0.14	0.08	0.09
T1	0.06	0.12	0.17	0.14	0.01
T2	0.08	0.05	0.02	0.02	0.01
T3	0.13	0.07	0.11	0.05	0.01
T4	0.08	0.15	0.10	0.11	0.01
T5	0.06	0.03	0.07	0.07	0.03
T6	0.17	0.16	0.10	0.11	0.01
T7	0.25	0.15	0.12	0.11	0.01
T8	0.00	0.03	0.01	0.01	0.01
Total	1	1	1	1	1

Table 10
Cross-table, knowledge levels cross utterers. Percentages are calculated row-wise, in order to analyze which knowledge level each participant is mostly using.

Knowledge Level Utterer	1	2	3	4	5	Total
F1	0.01	0.24	0.14	0.41	0.20	1
F2	0.05	0.40	0.28	0.23	0.05	1
Average for teachers	0.05	0.45	0.21	0.27	0.02	

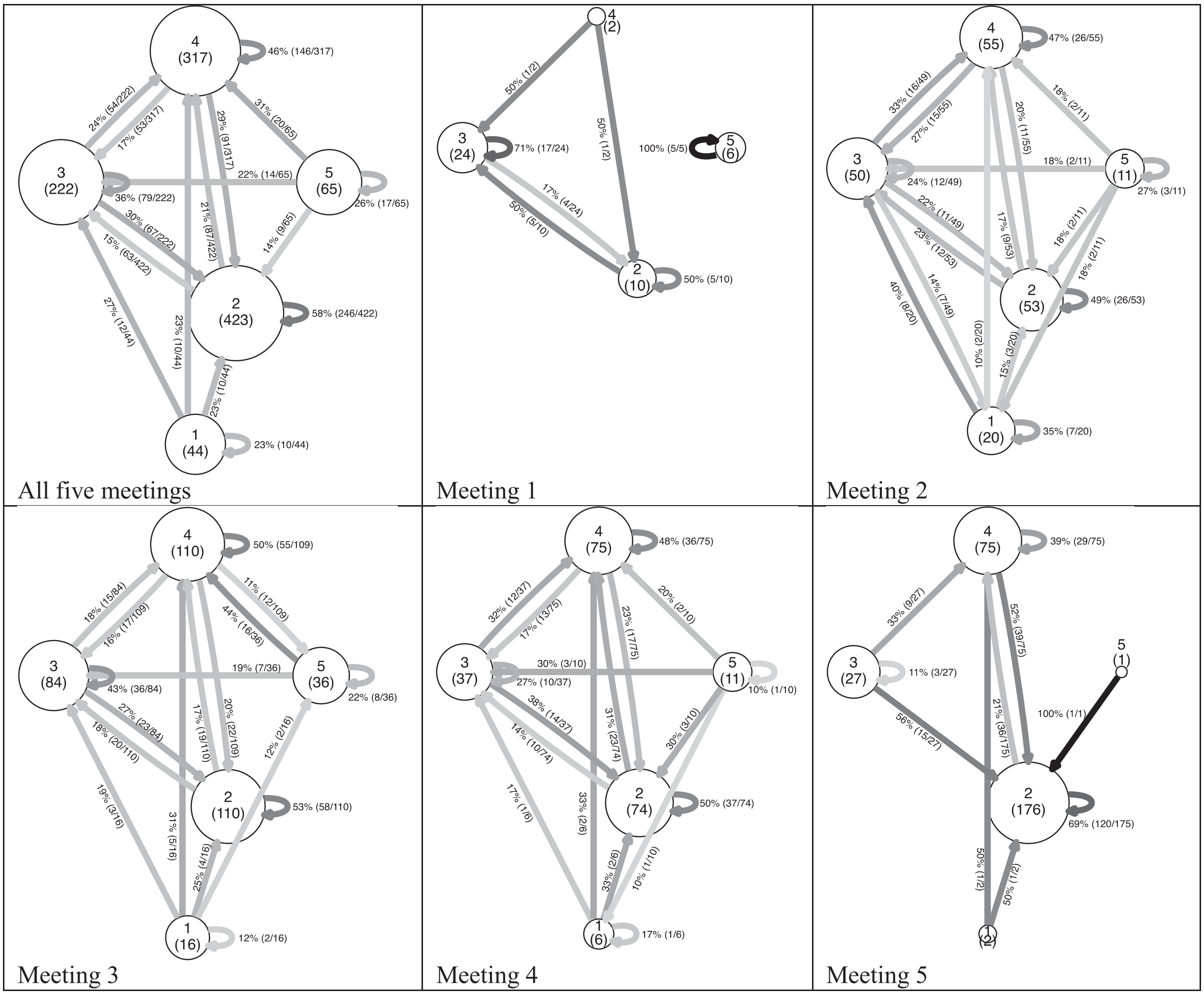

Fig. 5. Markov chains of knowledge level.

this meeting are to have a round-table discussion, with each teacher expressing their doubts, questions or difficulties related to PS. This result shows that teachers enter into the questioning. Meeting 1 also contains knowledge at level 2: teachers support their reasoning based on their classroom observations. This meeting ends with knowledge level 5: facilitators introduce PS teaching resources and comment on them at a general level. Thus, they use decontextualized knowledge about PS teaching from curricula and textbooks.

Meeting 2 deals with resources and the choice of task for the research lesson. This meeting is set up at levels 2, 3 and 4: for example, after an utterance expressing doubt or questioning (level 3), the next utterance has a high probability of containing a justification by means of contextualized explicit knowledge (level 4, 33%) or an observation or testimony (level 2, 56%). This means that the discussion at micro-level does not remain at the doubt or questioning level, as in the following extract (Table 12). In this extract, T6 and T4 compare two problems (problems B and C, which are not relevant to our analysis) with *The Fir Trees* problem (see Appendix). They express knowledge of mathematical problems at the level of questioning: are these three problems similar; do they require students to mobilize a great deal of knowledge? Then they deal with the resolution of the problem, they use knowledge of mathematical PS (do) at level 2. T5 summarizes T6 and T4′s words. T5 uses knowledge of mathematical problems at level 4 because he analyzes the

Table 11
Extract of meeting 2 (44:12–44:49).

Time	Utterer	Transcript	LSDA	MKTPS	Level
44:12	T4	Yeah, but it's infuriating, but during the [external state test], for the big problems, you can't tell them "it's like in this other problem". It [solving a problem] would be: it's a bit like this one, then like that one, then like that one … but all together!	RJ	K. of instructional practices for PS	3
44:25	F1	Thank you, teacher!	Humor	–	
44:28	T4	No but, I mean…	PA	–	
44:29	T6	If only they remember these…	RH	K. of students as mathematical problem solvers	1
44:31	T4	As far as they remember, that's it! Frankly, a thing that we do with my colleague, it is that, in grade 4, we take –but I think like everybody– we take some problems from the [previous year's state test], we give them to the students to solve, so that they prepare themselves psychologically, so that they know what fate has in store for them! Frankly, these are not like the small problems…	RJ	K. of instructional practices for PS	2

Table 12
Extract of meeting 2 (45:14–46:31).

Time	Utterer	Transcript	LSDA	MKTPS	Level
45:07	T6	[*The Fir Trees,* see Appendix] here, it's a bit like in [problem B] or. it's the same kind of problem, right?	QC	K. of math. problems	3
45:14	T4	No, [problem B] is worse. in [problem B] you have to change, you know, to put everything…well.	RJ	K. of math. problems	3
45:30	T6	[In problem B] you have to find the price of one balloon, and then you have to do…	RJ	K. of math. PS (do)	2
45:32	T4	Yeah, or the price of x balloons, and then it's. […]	RJ	K. of math. PS (do)	2
45:54	T5	But, in the same direction, this one [*The Fir Trees*] with the "soil", the trees, the advantage. well, it's not an advantage, but I find that what's interesting about this one, is that it not only has to do with money, but they also understand the liters, and then the half. So compared to, well this one [shows the chart with problem C], it's interesting too, […]. Whereas if we start with either the distance or with the measure or with the quantities, uh. it's going to be an additional difficulty that would have been interesting to explore too.	PE	K. of math. problems	4

mathematics involved in these problems and compares these problems. This extract points out the dynamics between teachers in the transitions between levels 2, 3 and 4.

Meeting 2 is also marked by the presence of knowledge level 1. Teachers dare to affirm the debatable representations of their students or the students' learning (see Table 11).

The discussions of meeting 3, that deal with teaching resources, are not marked by specific knowledge levels, but by several levels (level 2, level 4 and level 5).

The planning of the research lesson (meeting 4) is mainly set at level 2 and level 4. The discourse is located at knowledge level 2 because teachers prepare the research lesson from their own classroom observations. The discourse is also located at the level of contextualized explicit knowledge because, often, speakers argue and justify their didactical choices, in order to convince their colleagues during the preparation of the research lesson.

The knowledge level in meeting 5, the post-lesson meeting, is mainly set at level 2. This finding can be explained by the nature of the LS in the post-lesson discussion. During this phase, teachers are asked to argue and justify their comments, based on their observations of the students during the research lesson. Therefore, in a given utterance, the knowledge is most often at the observation and testimony level.

This analysis of knowledge levels highlights elements to help answer the research questions "Which knowledge levels are used during each LS meeting? Do they evolve?" The most prevalent knowledge levels by LS meeting should be explained by the specific objectives of the LS phase, as for the MKTPS. This analysis of knowledge levels also points out a dynamic between the main knowledge levels 2, 3 and 4.

To summarize, the previous sections point out the dynamics of the interactions for MKTPS and the knowledge levels. In order to grasp the construction of MKTPS during LS meetings, we continue this analysis, demonstrating the different ways in which this can be expressed in the course of the dialogue. More precisely, the next section focuses on the dynamics of the interactions for the LS dialogue analysis.

4.3. LS dialogue analysis

4.3.1. Markov chains of LS dialogue analysis

This section presents certain elements of the findings for research question 3: "What are the dynamics of the interactions of LS

Dialogue Analysis (LSDA) at the micro-level of communicative acts?”.

The utterances coded as answer or develop (R see Table 3) are the most represented across all LS meetings (752 utterances). In particular (Fig. 7), the speakers are asked to provide a justification, to go further or to develop their ideas (503 utterances), so as to provide clarification (207 utterances) or develop a hypothesis (42 utterances).

The Markov chain (Fig. 6) points out that the probability of selecting an utterance relating to answer or development is 45% for an utterance to position or coordinate, 74% for an utterance to arouse development or reasoning, 39% for an utterance to guide, 31% for an utterance to express or invite new ideas and 20% for a humorous utterance. This analysis points out that in a LS process, speakers are required to develop and to justify their own ideas.

This Markov chain shows other local dynamics in transitions between LSDA clusters: for example, after a position or coordinate utterance (P), the speaker answers or develops (R, 45%), guides (G, 11%) or positions or coordinates again (P, 29%). After an utterance to arouse development or reasoning (Q), the speaker answers or develops (R, 74%). After guidance, the speaker positions or co-ordinates (P, 24%), answers or develops (R, 39%) or arouses development or reasoning (Q, 10%).

After a new idea (E), the speaker positions or coordinates (P, 32%), answers or develops (R, 31%), arouses development or reasoning (Q, 16%) or guides (G, 10%). In particular, the Markov chain for LSDA codes (Fig. 7) allows us to refine our analysis: expressing new ideas (EE) comes after guidance (proposing an action, questioning (GA, 15%) or encouragement of dialogue (GG, 17%). After approving (PA), the speaker mostly answers or develops, primarily to provide a justification (RJ, 46%).

These examples show how the Markov chains (Fig. 6 and Fig. 7) allow us to point out dynamics in the interactions between speakers at the micro-level of communicative acts.

4.3.2. *Characterization of facilitators' and teachers' profiles for LS dialogue analysis*

The following section presents an analysis of the cross-tables of the LSDA clusters of utterers. It allows us to deal with the element of research question 3, related to facilitators and teachers: “Which types of interactions during LS meetings characterized the facilitators compared to those of teachers?”.

The facilitators mainly adopt a guidance (G) role during the LS meetings (88%, see Table 13). The profiles of the two facilitators are

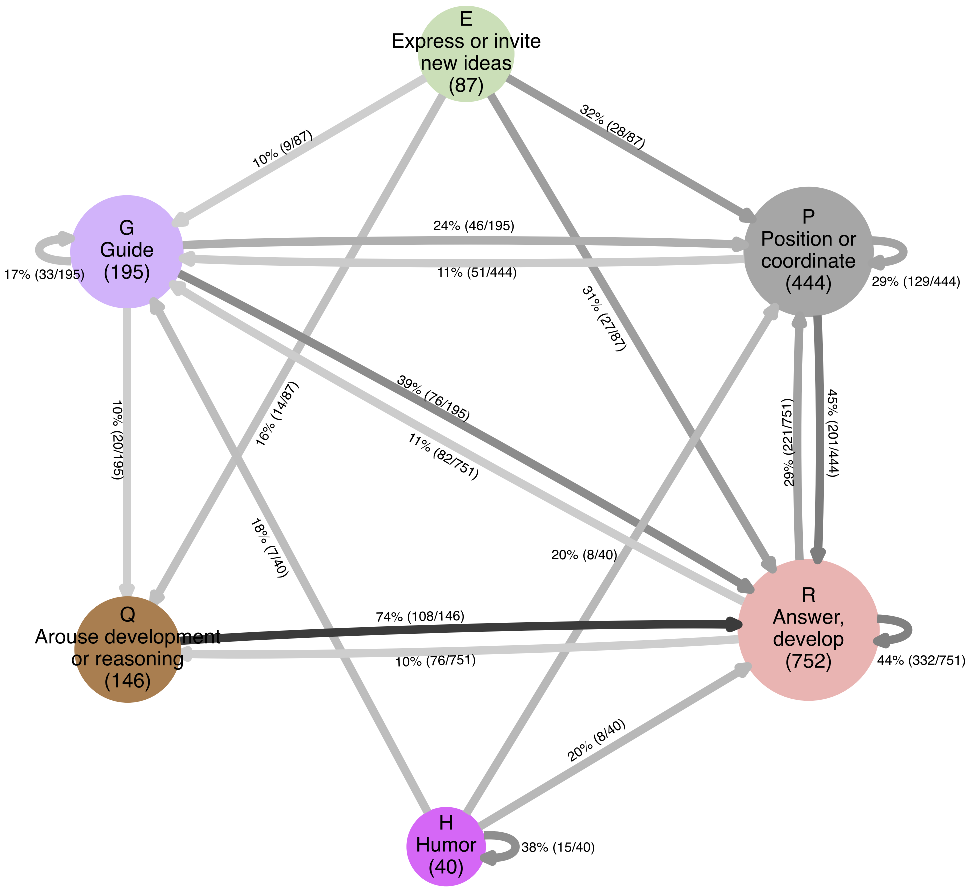

Fig. 6. Markov chain for the categories of LSDA for all LS meetings.

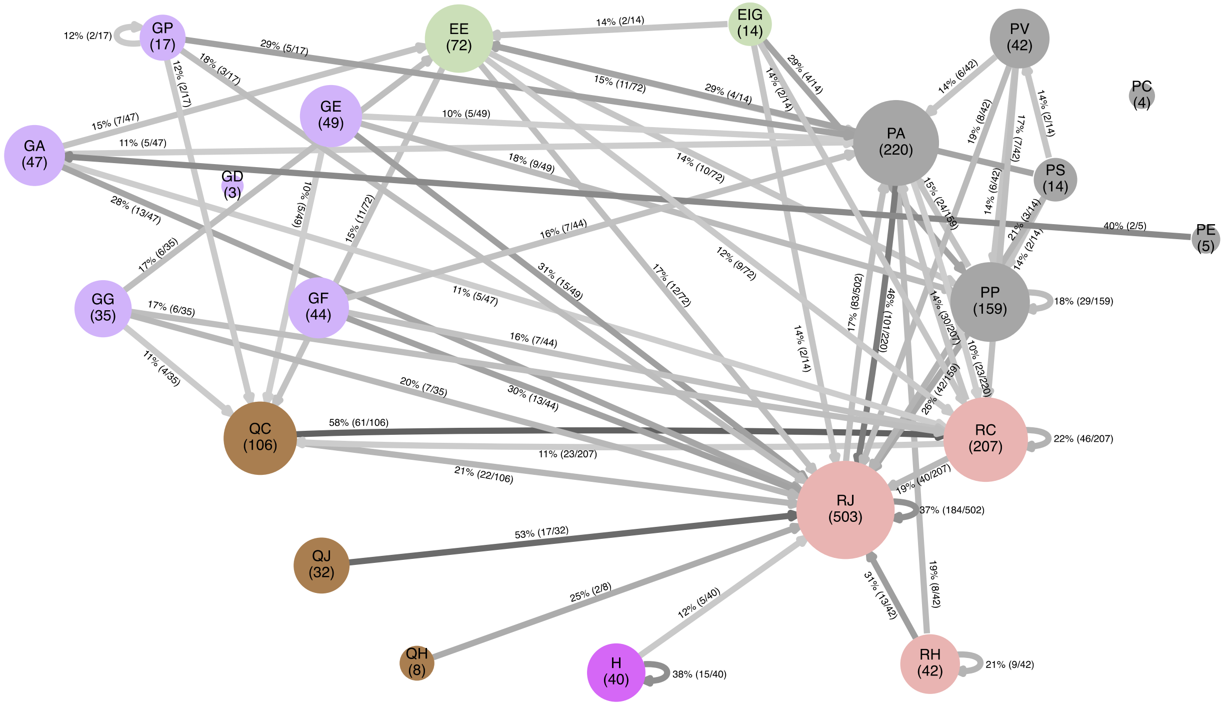

Fig. 7. Markov chain for LSDA codes for all LS meetings.

different. F1, who is a mathematics education researcher, mainly assumes the guidance (G) role during LS meetings (64%); in particular, he expresses an expert or authoritative perspective (GE, 98% by comparison with 2% for F2 see Table 14), he discusses the speech (GD, 67%) and he proposes or projects an action into the future (GP, 47%).

F1 positions or coordinates more than teachers (P, 17% as opposed to 9% see Table 13). In comparison with F2, he proposes more consensus (PC, 75% see Table 15) than F2 (25%) and synthesizes more ideas (PS, 64%) than F2 (14%).

F1 arouses development or reasoning (Q, 12% by comparison with 8% on average per teacher, see Table 13), and he answers and develops (R, 17% compared to 9% on average per teacher).

The new ideas express or invite new ideas (E), and are introduced by both teachers and facilitators. However, 80% of the total number of new ideas are introduced by the teachers and 20% by the facilitators. All teachers bring new ideas (between 5% and 16%, see Table 13).

F2, who is an experienced teacher in the LS process, shares a similar profile of interactions with other teachers, except for guidance and questioning. She guides (G, 24% by comparison with 2% on average per teacher, see Table 13) and arouses development or reasoning (Q, 23% more than teachers, who on average scored 8%).

To summarize, the two facilitators share two characteristics in their interactions: they guide (G), position or coordinate (P), arouse development or reasoning (Q) and answer and develop (R) more on average than teachers.

The following table (Table 16) points out the percentage of each type of interaction per speaker: 54% of teachers' interactions aim to answer and develop (R) and 27% position and coordinate (P); 37% of facilitators' interactions aim to answer and develop (R), 28% guide (G) and 20% position or coordinate (P). When comparing facilitators' and teachers' interactions, facilitators guide more (28% of facilitators' interactions on average focusing on guidance) than teachers (2% on average). Teachers answer or develop (54% of teachers' interactions on average) and introduce new ideas (8% of teachers' interactions on average) to a greater extent than the facilitators (37% and 3%, respectively). Teachers and facilitators position and coordinate, and arouse development or reasoning to the same extent on average.

We answer this part of research question 3 "Which types of interactions during LS meetings characterized the facilitators compared to those of the teachers?" by noting that the two facilitators share similarities in their interactions: they guide, position or coordinate, arouse development or reasoning, answer and develop more on average than teachers. However, their profiles distinguish between the expression of an expert or authoritative perspective, the discussions on the speech, the proposition of consensus, the synthesis of a greater number of ideas and the proposition or project relating to an action in the future.

Table 13
Cross-table, LSDA clusters cross utterers.

LSDA cluster / Utterer	E	G	H	P	Q	R
F1	0.11	0.64	0.23	0.17	0.12	0.17
F2	0.08	0.24	0.03	0.09	0.23	0.11
T1	0.16	0.06	0.10	0.11	0.15	0.10
T2	0.06	0.00	0.00	0.04	0.04	0.04
T3	0.09	0.02	0.00	0.07	0.09	0.07
T4	0.10	0.01	0.08	0.12	0.02	0.13
T5	0.07	0.01	0.00	0.05	0.01	0.06
T6	0.13	0.01	0.18	0.11	0.09	0.15
T7	0.15	0.03	0.25	0.21	0.24	0.14
T8	0.05	0.00	0.00	0.00	0.01	0.02
Total	1	1	1	1	1	1
Average for facilitators	0.10	0.44	0.13	0.13	0.17	0.14
Average for teachers	0.10	0.02	0.08	0.09	0.08	0.09
Total for facilitators	0.20	0.88	0.25	0.26	0.35	0.28
Total for teachers	0.80	0.12	0.60	0.72	0.65	0.71

Table 14
Cross-table, guidance cross utterers.

LSDA code / Utterer	GA	GD	GE	GF	GG	GP
F1	0.60	0.67	0.98	0.52	0.43	0.47
F2	0.19	0.33	0.02	0.41	0.46	0.12
Total for teachers	0.21	0.00	0.00	0.07	0.11	0.41
Total	1	1	1	1	1	1

Table 15
Cross-table, position and coordinate cross utterer.

LSDA code / Utterer	PA	PC	PE	PP	PS	PV
F1	0.13	0.75	0.20	0.16	0.64	0.19
F2	0.09	0.25	0.40	0.07	0.14	0.14
Total for teachers	0.78	0.00	0.40	0.77	0.14	0.67
Total	1	1	1	1	1	1

Table 16
Cross-table, LSDA clusters cross facilitators and teachers (% per line).

LSDA cluster / Utterer	E	G	H	P	Q	R	Total
F1	0.03	0.34	0.02	0.20	0.05	0.35	1
F2	0.03	0.22	0.00	0.20	0.16	0.38	1
Average of facilitators	0.03	0.28	0.01	0.20	0.10	0.37	1
Average of teachers	0.08	0.02	0.01	0.27	0.08	0.54	1

5. Discussion and conclusion

The results section has yielded some answers to our research questions (see Table 5) regarding MKTPS collective building, knowledge level evolution and the dynamic of interaction. In this section, we collate and synthesize these three aspects. We then focus on the facilitators' and teachers' characteristics as they appear in our analysis. Limits and perspectives are finally considered.

Regarding research question 1, the Markov chains for MKTPS allow for movement from a static to a dynamic viewpoint in relation to knowledge. Not only MKTPS specific to the teaching and learning of problem-solving are the most represented during our LS meetings, but they also pull the focus of the dialogue towards them: participants collectively use their mathematical knowledge for teaching and their PS content knowledge to focus on the knowledge of students as mathematical problem solvers and the knowledge of instructional practices for PS.

This dynamic vision is also perceptible between knowledge levels, bringing element to answer research question 2. On the one hand, incomplete knowledge, knowledge with a low degree of certainty or explicit questioning (level 3) expressed by one utterer tends to lead to explicit knowledge (level 4) or an observation (level 2). On the other hand, knowledge (implicit or explicit) expressed by one utterer often gives way to questioning by the next utterer. Generalized or decontextualized knowledge (level 5) is mainly expressed by facilitator 1 (F1), who has the role of knowledgeable other. This role is also shown in the dialogue by the vast majority of utterances, such as “express an expert or authoritative perspective” or “propose a consensus” and “synthesize ideas”, expressed by F1.

As for research question 3, the presence of two facilitators, with different backgrounds, allows facilitator 2 (F2) to have a more important role in the dialogue, in terms of arousing development or reasoning. Her identity as a teacher from the school is reflected in her dialogic role, close to the one of the teachers. Regarding the teachers’ role in the dialogue, they bring as many new ideas as the facilitators. Moreover, every teacher brings new ideas used in the following discourses, and certain dialogic roles are equally shared: teachers and facilitators position and coordinate, arouse development or reasoning in similar proportions. This dialogic analysis points out how each teacher and facilitator contributes to the LS process and brings elements of dialogic fundaments to the claim of many LS researchers: “Something striking about lesson study is the coequal status of participants. Despite the differences in experience that may occur in a group of teachers, it is assumed that every member will have something important to contribute to lesson study” (Lewis & Hurd, 2011, p. 94).

The interplay between the three aspects of our analysis, as presented in this paper, has significant potential, but also many limitations or even a series of limitations. The first series is linked to the time-consuming nature of our coding and analysis process. This resulted in the restriction of the width and depth of our research. The width should be expanded by coding and analyzing three more meetings (planning of the second research lesson and its post lesson discussion, conclusive discussion and preparation of next cycle) and a second cycle regarding PS teaching (same group, over six meetings). The depth should also be expanded by digging more deeply into the cross-tables (for the interplay between the three aspects) and the Markov chains (for the dynamic aspect). In fact, we will need another statistical model, allowing to cross knowledge with knowledge levels and interactions, in order to explain knowledge building within the chronology of the dialogue.

The second series of limitations are linked to our analytic tools. The dynamic vision brought by the Markov chains is limited to the succession of two utterances. However, in order to grasp the construction of knowledge with the same type of mixed method that we used, we would need to consider longer chains and successions of communicative acts (utterances). We have been exploring one way of doing so in accordance with Littleton and Mercer (2013), Dudley (2013) and Kershner et al. (2020), moving from the micro-level point of view of communicative acts characterized by LSDA, to the meso-point of view of communicative events characterized by types of speech. We characterize communicative acts in terms of “1. cumulative talk, 2. qualifying or disputational talk, 3. exploratory talk, 4. structuring conversation, 5. managing understanding” (Dudley, 2013, p. 110) and link these types of talk to MKTPS and to knowledge levels (Clivaz et al., 2023).

The third type of limitation is linked to our choice of looking at the collective aspect of knowledge in dialogue. Our coded data could be used to look at each individual participant and to analyze the way in which their participation in the dialogue is related to the individual building of knowledge. This promising perspective aims to better understand how the professional knowledge development of teachers occurs in a LS process.

Lastly, we are aware that this case study is only linked to one particular group and that individual characteristics influence the results. However, despite these limitations, the mixed method we used allowed for a dynamic and dialogic view of the collective construction of MKTPS during a LS process. The particularity of both LS and MKTPS shown in our results sheds light on the way in which LS and PS teaching are “two wheels of a cart” (Fujii, 2018) and how teachers’ professional knowledge is developed in a dialogic process during LS.

Funding

This research has been funded by Lausanne University of Teacher Education.

CRediT authorship contribution statement

Stéphane Clivaz: Supervision, Conceptualization, Data curation, Software, Investigation, Methodology, Project administration, Visualization, Writing. **Valérie Batteau:** Conceptualization, Formal analysis, Methodology, Writing. **Jean-Philippe Pellet:** Data curation, Formal analysis, Software, Methodology, Visualization. **Luc-Olivier Bünzli:** Conceptualization, Data curation, Formal analysis. **Audrey Daina:** Conceptualization, Formal analysis, Methodology, Writing. **Sara Presutti:** Conceptualization, Data curation, Formal analysis, Software, Investigation.

Conflict of interest

None.

Data Availability

The data that has been used is confidential.

Appendix. the problem, taken from State of Vaud (Switzerland) state-wide test 2015, our translation

The Fir Trees

Sam wants to plant fir trees in his nursery.
He has to buy 2 kinds of soil:

– universal soil ;
– special soil for fir trees.

In total, he needs 400 litres of soil, half of each kind.

How much will Sam have to pay in total?

.

References

Ball, D. L. (2017). Uncovering the special mathematical work of teaching. In G. Kaiser (Ed.), *Proceedings of the 13th International Congress on Mathematical Education, ICME-13* (pp. 11–34). Springer International Publishing.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education, 59*(5), 389–407. https://doi.org/10.1177/0022487108324554

Bikner-Ahsbahs, A., & Prediger, S. (2010). Networking of theories—An approach for exploiting the diversity of theoretical approaches. In S. Lerman (Ed.), *Theories of mathematics education* (pp. 483–506). Springer.

Bjuland, R., & Helgevold, N. (2018). Dialogic processes that enable student teachers' learning about pupil learning in mentoring conversations in a lesson study field practice. *Teaching and Teacher Education, 70*, 246–254. https://doi.org/10.1016/j.tate.2017.11.026

Borko, H., & Potari, D. (Eds.). (2020). *ICMI Study 25 Conference Proceedings: Teachers of Mathematics Working and Learning in Collaborative Groups*. University of Lisbon. ⟨http://icmistudy25.ie.ulisboa.pt/wp-content/uploads/2020/05/ICMIStudy25Proceedings.pdf⟩.

CEDiR. (n.d.). *CEDiR: Cambridge Educational Dialogue Research*. ⟨https://www.educ.cam.ac.uk/research/groups/cedir//⟩.

Chapman, O. (2015). Mathematics teachers' knowledge for teaching problem solving. *LUMAT, 3*(1), 19–36. ⟨https://pdfs.semanticscholar.org/327b/973f1bccfc73f888aad4532f4c3b98f16a24.pdf⟩.

Clivaz, S., & Clerc-Georgy, A. (2020). Facilitators' roles in lesson study: from leading the group to doing with the group. In A. Murata, & C. Lee (Eds.), *Stepping up lesson study: An educator's guide to deeper learning* (pp. 86–93). Routledge.

Clivaz, S., Daina, A., Batteau, V., Presutti, S., & Bünzli, L.-O. (2023). How do dialogic interactions contribute to the construction of teachers' mathematical problem-solving knowledge? Construction of a conceptual framework. *International Journal for Lesson & Learning Studies*. https://doi.org/10.1108/IJLLS-03-2022-0031

Clivaz, S., & Ni Shuilleabhain, A. (2019). What knowledge do teachers use in lesson study? A focus on mathematical knowledge for teaching and levels of teacher activity. In R. Huang, A. Takahashi, & J. P. da Ponte (Eds.), *Theory and practice of lesson study in mathematics: An international perspective* (pp. 419–440). Springer. https://doi.org/10.1007/978-3-030-04031-4_20.

Clivaz, S., & Takahashi, A. (2018). Mathematics lesson study around the world: Conclusions and looking ahead. In M. Quaresma, C. Winsløw, S. Clivaz, J. P. da Ponte, A. Ni Shuilleabhain, & A. Takahashi (Eds.), *Mathematics lesson study around the world: Theoretical and methodological issues* (pp. 153–164). Springer. https://doi.org/10.1007/978-3-319-75696-7.

Dudley, P. (2013). Teacher learning in lesson study: What interaction-level discourse analysis revealed about how teachers utilised imagination, tacit knowledge of teaching and fresh evidence of pupils learning, to develop practice knowledge and so enhance their pupils' learning. *Teaching and Teacher Education, 34*(0), 107–121. https://doi.org/10.1016/j.tate.2013.04.006

Fujii, T. (2018). Lesson study and teaching mathematics trough problem solving: The two wheels of a cart. In M. Quaresma, C. Winsløw, S. Clivaz, J. P. da Ponte, A. Ni Shuilleabhain, A. Takahashi, & T. Fujii (Eds.), *Mathematics lesson study around the world: Theoretical and methodological issues* (pp. 1–22). Springer.

Gagniuc, P. A. (2017). *Markov chains: From theory to implementation and experimentation*. John Wiley & Sons.

Hennessy, S., Rojas-Drummond, S., Higham, R., Márquez, A. M., Maine, F., Ríos, R. M., García-Carrión, R., Torreblanca, O., & Barrera, M. J. (2016). Developing a coding scheme for analysing classroom dialogue across educational contexts. *Learning, Culture and Social Interaction, 9*, 16–44. https://doi.org/10.1016/j.lcsi.2015.12.001

Theory and practices of lesson study in mathematics: An international perspective. In Huang, R., Takahashi, A., & da Ponte, J. P. (Eds.), *ZDM*, (2019). Springer.

Kershner, R., Hennessy, S., Wegerif, R., & Ahmed, A. (2020). *Research methods for educational dialogue*. Bloomsbury Academic.

Lewis, C. (2002). Lesson study: A handbook of teacher-led instructional change. *Research for Better Schools*.

Lewis, C. (2015). What is improvement science? Do we need it in education? *Educational Researcher, 44*(1), 54–61.

Lewis, C., Friedkin, S., Emerson, K., Henn, L., & Goldsmith, L. (2019). How does lesson study work? Toward a theory of lesson study process and impact. In R. Huang, A. Takahashi, & J. P. da Ponte (Eds.), *Theory and practice of lesson study in mathematics: An international perspective* (pp. 13–37). Springer International Publishing. https://doi.org/10.1007/978-3-030-04031-4_2.

Lewis, C., & Hurd, J. (2011). *Lesson study step by step: How teacher learning communities improve instruction*. Heinemann.

Lewis, C., Perry, R., & Murata, A. (2006). How should research contribute to instructional improvement? The case of lesson study. *Educational Researcher, 35*(3), 3–14. https://doi.org/10.2307/3700102

Lewis, J. M. (2016). Learning to lead, leading to learn: How facilitators learn to lead lesson study. *ZDM*, 1–14. https://doi.org/10.1007/s11858-015-0753-9

Liljedahl, P., Santos-Trigo, M., Malaspina, U., & Bruder, R. (Eds.). (2016). *Problem solving in mathematics education*. Springer International Publishing. https://doi.org/10.1007/978-3-319-40730-2_1.

Littleton, K., & Mercer, N. (2013). *Interthinking: Putting talk to work*. Taylor & Francis. 〈https://books.google.com.my/books?id=C26pAgAAQBAJ〉.

Mayer, R. E., & Wittrock, M. C. (2006). Problem solving. In P. A. Alexander, & P. H. Winne (Eds.), *Handbook of educational psychology* (Vol. 2, pp. 287–303). Erlbaum.

Ni Shuilleabhain, A., & Clivaz, S. (2017). Analyzing teacher learning in lesson study: Mathematical knowledge for teaching and levels of teacher activity. *Quadrante, 26*(2), 99–125.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: first steps towards a conceptual framework. *ZDM, 40*(2), 165–178. https://doi.org/10.1007/s11858-008-0086-z

Quaresma, M., Winsløw, C., Clivaz, S., Ponte, J. P., Ni Shuilleabhain, A., & Takahashi, A. (Eds.). (2018). *Mathematics lesson study around the world: Theoretical and methodological issues*. Springer. https://doi.org/10.1007/978-3-319-75696-7.

SEDA. (n.d.). *A Tool for Analysing Dialogic Interactions in Classrooms.* 〈http://www.educ.cam.ac.uk/research/projects/analysingdialogue/〉.

Seino, T., & Foster, C. (2020). Analysis of the final comments provided by a knowledgeable other in lesson study. *Journal of Mathematics Teacher Education*. https://doi.org/10.1007/s10857-020-09468-y

Stigler, J., & Hiebert, J. (1999). *The teaching gap. Best ideas from the worlds teachers for improving education in the classroom*. The Free Press.

Takahashi, A. (2014). The role of the knowledgeable other in lesson study: Examining the final comments of experienced lesson study practitioners. *Mathematics Teacher Education and Development, 16*, 4–21.

TIMSS Video. (n.d.). *The TIMSS Video Study.* 〈http://www.timssvideo.com〉.

Vermunt, J. D., Vrikki, M., van Halem, N., Warwick, P., & Mercer, N. (2019). The impact of lesson study professional development on the quality of teacher learning. *Teaching and Teacher Education, 81*, 61–73. https://doi.org/10.1016/j.tate.2019.02.009

Vrikki, M., Warwick, P., Vermunt, J. D., Mercer, N., & Van Halem, N. (2017). Teacher learning in the context of lesson study: A video-based analysis of teacher discussions. *Teaching and Teacher Education, 61*, 211–224. https://doi.org/10.1016/j.tate.2016.10.014

Vygotsky, L. S. (1962). *Thought and language*. MIT Press.

Vygotsky, L. S. (1978). *Mind in society*. Harvard University Press.

Warwick, P., Vrikki, M., Vermunt, J. D., Mercer, N., & van Halem, N. (2016). Connecting observations of student and teacher learning: an examination of dialogic processes in Lesson Study discussions in mathematics. *ZDM, 48*(4), 1–15. https://doi.org/10.1007/s11858-015-0750-z

Watanabe, T., Takahashi, A., & Yoshida, M. (2008). Kyozaikenkyu: A critical step for conducting effective lesson study and beyond. In F. Arbaugh, & P. M. Taylor (Eds.), *Inquiry into Mathematics Teacher Education* (pp. 131–142). Association of Mathematics Teacher Educators.

Watanabe, T., & Wang-Iverson, P. (2005). The role of knowledgeable others. In P. Wang-Iverson, & M. Yoshida (Eds.), *Building our understanding of lesson study* (pp. 85–92). Research for Better Schools.

Yoshida, M. (2012). Mathematics lesson study in the United States. *International Journal for Lesson and Learning Studies, 1*(2), 140–152. https://doi.org/10.1108/20468251211224181