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Recent years have been marked by a growing research interest in students with Mathematical Learning Difficulties (MLD, acronym which denotes specific and/or severe difficulties in mathematics). Most research on MLD has focused almost exclusively on the arithmetic domain, but in recent years, research has begun to taking into consideration other mathematical domains, for example algebra. The present research aims at answering the following research question: What is the role of examples in algebraic thinking for students with MLD? Using Balacheff's typology of proofs, different types of examples are identified: naïve empiricism, crucial experiment, example to spot the regularity and generic example. These examples can be used as tools for observing algebraic thinking in students with MLD and they support the occurrence of algebraic thinking in this population.

Keywords: MLD, mathematical learning difficulties, mathematical learning disabilities, early algebra, role of examples.

Introduction and literature review

Recent years have been marked by a growing research interest in *Mathematical Learning Difficulties* (MLD). This interest, which until a few years ago belonged mainly to the psychological domain, now also concerns mathematics education. An example of this growing interest is the recent creation of *TGW25 Inclusive Mathematics Education – Challenges for Students with Special Needs* for CERME11 in 2019.

The relative youth of this field is reflected in a lack of unanimity on the meaning of the acronym MLD. Some researchers speak of *Mathematical Learning Difficulties*, others of *Disabilities* and others of *Disorders* (Baccaglini et al., 2020). The acronym is therefore used in mathematics education in a polysemic way and to identify different populations (Deruaz et al., 2020; Lewis & Fisher, 2016; Scherer et al., 2016). According to Deruaz et al. (2020), it can refer to students who have been diagnosed with a learning disorder specific to mathematics through a standardised test, usually psychological, and through defined criteria (such as cutoff, etc.). The same acronym can be found to refer to students who have been diagnosed with another learning disorder, not specific to mathematics, but which may have an impact on their learning (e.g., dyslexia or dyspraxia). The term MLD is also used to refer to students who have severe difficulties in mathematics without ever having been diagnosed. The latter category is designated through non-standardised tests, or through teacher assessment.

Despite the wide variety of definitions, they have in common the focus on students with specific and/or severe difficulties in mathematics (Deruaz et al. 2020). This "inclusive" vision of MLD which is not necessarily linked to a medical certificate attesting the difficulties of the students seems to be the most appropriate for mathematics education. Indeed, this research field aims to take charge of all students and their difficulties, regardless of whether or not a diagnosis has been obtained.

Most research on MLD has focused almost exclusively on the arithmetic domain, on number sense and basic arithmetic calculations (Deruaz. et al., 2020; Lewis & Fisher, 2016). In recent years, research has begun to broaden its scope, taking into consideration other mathematical domains, for example equations are included by Karagiannakis et al. (2016) in a battery of tasks designed to detect the causes of difficulties for students with MLD. Furthermore, recent studies in cognitive science have shown that learning difficulties in mathematics are heterogeneous (Fias et al., 2013) and affect several aspects of mathematical skills (Kaufmann et al., 2013). Although there are early signs of interest in other areas of mathematics, these are very rare. This finding shows the need for further research concerning students with MLD, that considers other areas of mathematics. And, as Lewis and Fisher (2016, p.365) say, “in particular, it is critically important that researchers begin to explore MLD in algebra, given its role as an educational gatekeeper”.

The last decades of research in mathematics education on algebra have been marked by an interest in *early algebra*, a specific area of teaching identified as a bridge between arithmetic and algebra (Malara and Navarra, 2018). It is a meta-discipline that links arithmetic and algebra and can use mathematical tasks and problems traditionally presented in the arithmetic domain to highlight algebraic processes and algebraic reasoning necessary for a good understanding of algebra (in the traditional sense, Malara and Navarra, 2018). This approach favours the development of a certain way of thinking, *algebraic thinking*, which does not necessarily need the standard algebraic symbolism to be addressed and which is necessary for a proper learning of traditional algebra (Kieran et al., 2016). Malara and Navarra (2018) identify some main language constructs that are fundamental in order to generate new ways to see arithmetic and thus algebraic thinking. One of them is argumentation. Argumentation and its verbalisation are crucial in the approach to generalisation and early algebra. In fact, it fosters students to explicit ideas and procedures of which they were not fully aware before trying to express them. Argumentation and justification make it possible to make explicit an algebraic reasoning that would otherwise remain implicit.

Early algebra therefore seems to be a mathematical domain that is particularly well suited to research about students with MLD because it allows the research to be taken up where it has been left off, at arithmetic, and to bridge the gap with the new mathematical discipline, algebra. Furthermore, it is ideal with students with MLD because it allows algebraic concepts to be tackled without standard algebraic symbolism (which not all students with MLD encounter in their schooling). Although this topic is of great scientific interest, we are currently unaware of any research publications concerning students with MLD in early algebra.

Theoretical framework and research question

The literature review described above led us to become interested in the behaviour of students with MLD in early algebra, wanting to tackle the problem of how we can describe the algebraic thinking of students with MLD. In particular, as we see in the previous section, argumentation and justification are fundamental for algebraic thinking. This consideration leads us to the research question: *What is the role of examples in algebraic thinking for students with MLD?* Our hypothesis is that examples are an opportunity for producing argumentation and justifications and thus showing algebraic thinking of students with MLD. Focusing on examples, we can observe students with MLD using algebraic thinking. On the one hand, therefore, examples serve to study the algebraic reasoning of students with MLD. On the other hand, students with MLD are a good population to study the role of examples in mathematical reasoning since they use examples often. It is important to note that the present research focuses on studying the reasoning of students with MLD, without having the ambition to propose a teaching intervention. This could be an idea for future research that would build on the results of current research to construct the intervention.

To answer the research question, we relied on the theoretical framework of Balacheff's (1987), who created a typology for proofs. The typology offers a classification on the basis of the knowledge involved and the nature of the underlying rationality. From this perspective, the proof is understood as an explanation accepted at a certain point in time by a certain community. There are two main types of proof: pragmatic proofs and intellectual proofs. Pragmatic proofs are action-related and carried out by the students themselves to establish the truth of a certain proposition. If access to this realisation is not possible and the action must be abandoned, we speak of intellectual proofs.

1. *Naïve empiricism* is the first stage of pragmatic proofs. It occurs when the validity of a statement is proved from one or a small number of cases.

2. The *crucial experiment* proves a statement by presenting an example that the student recognises as being as non-specific as possible. If the proposition is true in this case, then it must necessarily be always true. Crucial experiment, which remains a pragmatic proof, differs from naïve empiricism in that the generality of the proposition is taken into account and made explicit.

3. A *generic example* lies on the borderline between pragmatic and intellectual proofs. A proposition can be proved by means of the generic example when a specific case is not treated in its particularity but as representative of a certain family of objects with an argument that can be extended to a whole class of objects. This type of proof consists of proving the validity of an assertion by performing operations or transformations on a particular case, but at the same time use is made of the properties and structure that characterise the class that this particular case represents.

4. *The thought experiment* allows proving by internalising the action and detaching it from its concretisation on a particular representative. By remaining linked to anecdotal temporality, it abandons the treatment of a particular case, as was the case for the generic example.

This typology should not be understood as a tool to assign each student a possible level of knowledge or to identify the cognitive level they are at. It is not a set of successive stages that students must reach in a given order, it is simply a tool for describing students' actions in a certain context in a given mathematical task.

Method

Context and participants

The data presented in this paper has been selected from more extensive research in which more students (19 in total) and more mathematical problems (8 in total) were taken into account.

Data were collected in the canton of Vaud, in the French-speaking part of Switzerland. In this region, interest in school inclusion has grown in recent years. This is evident for example from the creation of the *Concept 360°* (DFJC, 2019), a project which aims to establish the principles of a school that responds to the specific needs of all students. In spite of this growing interest, school organisation is divided into different types of schools (ordinary schools and specialised schools) or school levels depending on pupils' abilities and grades.

The participants were selected based on their severe and/or persistent difficulties in mathematics, as defined by Deruaz. et al. (2020) and described in the previous paragraphs.

In particular, for the research presented in these pages, the students belonged to different classes of lower secondary school (7^{th}-9^{th} grade in Switzerland, 12-14 years old) and have different profiles.

1. Student A (9^{th} grade) is enrolled in an ordinary school, in the level for students with the best grades. Student A has good results in all subjects except mathematics, in which she is considered to be in severe difficulty by her teacher.
2. Student B (8^{th} grade) is schooled in a special teaching class. Student B has a diagnosis of dysphasia and dyslexia and has severe difficulties in all subjects, including mathematics.
3. Student C (8^{th} grade) is enrolled in an ordinary school, in the level for students with low grades. Student C has a diagnosis of dyscalculia and dyspraxia.

Procedure

Data were collected through clinical interviews. This is a semi-directive, open-ended interview between a researcher and a student, whose aim is to encourage the manifestation and observation of mathematical thinking (Ginsburg, 1981). The student interviewed had the task of solving the assigned problem by explaining their procedure. The researcher intervened to ask questions requesting clarification of the procedure used or to unblock the situation in case of difficulty. The aim of the interview was to get the students to show a large number of examples and to progress in their mathematical reasoning. The researcher therefore tried to create the ideal contextual conditions for this objective, by relaying the students' statements to allow them to show their full potential, but without replacing them in finding the answer.

The interviews are filmed and the audio transcribed. The unit of analysis consisted of students' oral and written contributions. These have been analysed on the basis of the typology of proofs (Balacheff, 1987) with a particular focus on the use of examples in algebraic thinking: each sentence said by the students and each written production produced were read and, when relevant, categorised according to a category of the typology.

Mathematical problem

The interview was about solving the following mathematical problem: *If I add two odd numbers, do I always get an even number?*[1]

The chosen problem develops algebraic thinking as it allows us to work on a property of numbers (being odd or even) and on the structure of ℤ (there is a regular alternation of odd and even numbers) by providing evidence for the reasoning carried out. The chosen problem is particularly well suited to answering the research question because the understanding of the statement and the result are facilitated by the mediation of the examples, and examples are not provided directly from the statement but must be created independently by the student, according to his needs.

Results

The analysis through Balacheff's (1987) framework of the clinical interviews allowed us to identify different types of examples created by the students and used to solve the problem.

The first type of example, the *naïve empiricism*, is what the student produces right after starting to work on the problem, it is the first example which lets her begin tacking the problem. For instance, student B starts the problem in the following way:

Researcher:	Yes, it requires taking...
Student B:	Multiples of two.
Researcher:	Two odd numbers. If I take two odd numbers and add them together, do we get an even or an odd number?
Student B:	Yes… even… we can… if you do 3+3, it gives 6.

The examples enable the students to start reasoning and to approach the problem that otherwise would remain unreachable.

After the first example, students give other examples, and then other examples, until generality is taken into account and explicitly evoked. For example, student B continues his reasoning:

Researcher:	What do you think, if I add an even number plus another even number, how will the result be?
Student B:	Even.
Researcher:	How do you know?
Student B:	Well, if you add 12 plus 12 that's 24 and if you add 14 plus 14 that's 28 and it's always even, otherwise 4 plus 4 is 8, 6 plus 6 is 12, 8 plus 8 is 16, it's always even.

This list of examples ensures the generality of the statement. There are so many examples that, for the student, this is enough to support the generality of the statement. We call this the *crucial experiment*. The crucial experiment may also be given by a single example which in the view of the students is so unspecific that if the statement is true in that case, then it is always true. What is important for the crucial experiment is that the generality of the situation is evoked, in this aspect the crucial experiment is different form the naïve empiricism.

[1] The mathematical problem and transcriptions were translated into English by the author of the text from the original French version.

This iteration of examples leads the student to spot the regularity of the situation and to understand that this regularity has a motivation. The student evokes the fact that the regularity can be identified, understood and generalised. Student A says:

Researcher: We want to understand whether if we add two odd numbers, we always get an even number. Do you remember what an even number is, and an odd number?

Student A: Yes, of course I do. Well... by giving examples. For example, 3 plus 3 is 6. 3 plus 7 is 10. 3 plus 11 is 14. So yes, I think it will always be like that, because there is something logical behind it. In any case, I don't have an example that comes to mind where it wouldn't be possible.

Here the students take into consideration the generality of the situation and she does something more: the example let her see that the regularity has motivations and can be understood and explained. It is "logical", as the student says. The example generates in the student the need for regularity. With respect to Balacheff's (1987) typology, his example, *example to spot the regularity*, is not a new category but it is between the crucial experiment and the generic example. It has some characteristics of the pragmatic proofs (it is linked to the action on a limited number of particular cases) and some other characteristics of the intellectual proofs (not only is regularity evoked as in the crucial experiment, but the existence of a logical structure behind this regularity is also evoked).

Examples can also be used to generalise the regularity through a *generic example*. In this case, the example is not treated in its particularity, but as representative of a certain family of objects. The generic example uses the characteristic properties and structures of a certain class of objects by relying on one of its representatives for the implementation of the reasoning. For instance, student C writes $7 + 7 = 14 : 2 = 7$ (Figure 1) and says:

Researcher: Okay, so the question here is... When you add two numbers that are odd, is the result even or odd?

Student C: Yes, okay.

Researcher: Do you have any ideas?

Student C: Well, I could take, for example, 3 which is odd plus 3. That's 6, which is even because if you do 6 divided by 2, it's 3. Then if I take 5 plus 5 which is odd, it's 10. Which is also divided by 2. 7 plus 7, 14, which is also divided by 2. Well, yes, because each time you make an odd number plus another odd number, the same one (indicating 7 on his example), it gives this number (indicating 14) and this number is even because each time, you can divide it by 2 to give 7... well, to give the (indicating 7)…

Here the student is proving that every time that we add two times the same odd number, the result can be divided by 2, so it is even. He solves a particular case of the given problem. He uses the example "7 plus 7", to support a general reasoning and this particular example is necessary to produce a reasoning that otherwise he couldn't have had.

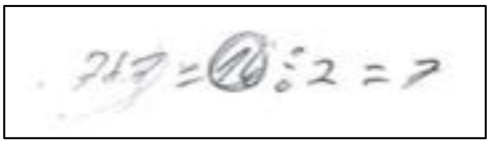

Figure 1: Generic example given by student C

It is interesting to note that this example falls into the generic example category because in the words used by student C we can find references to the particular case ("it gives *this* number"). And also, the gestures (e.g. "*indicating* 14") show that the reasoning is based on the particular case reported in Figure 1. A less thorough analysis could have suggested a thought experiment.

Discussion

The results provide some first answers to the research question: *What is the role of examples in algebraic thinking for students with MLD?*

First of all, the results show that different examples can be used for different purposes (Table 1). They can be used for the student herself, with an *internal purpose*. Examples are essential for the exploration phase: understanding the statement and getting an idea of the results, taking ownership of the problem, entering in reasoning (naïve empiricism), conjecturing (crucial experiment), to spot the regularity of the situation. Examples are also used for solving and proving the conjecture (generic example).

Table 1: The typology of examples in algebraic proofs

Naïve empiricism	Crucial experiment	Example to spot the regularity	Generic example
The first example which lets the students in the problem and in reasoning	The generality of the situation is evoked	The situation is recognised as regular	The example is used as representative of a family of objects

In addition to the internal purposes, examples can also be used in *interaction with others*, to communicate the results obtained.

Table 1 shows how the examples generated by the argumentation promote the generalisation of the mathematical problem by inducing a tendency for students to think algebraically.

Examples can have different status and different roles in algebraic thinking and proving; they can be used as tools for reasoning and for producing algebraic thinking. Examples are particularly important for students with severe difficulties as students with MLD because in this case they support students' thinking, algebraic thinking, which would not be possible without them.

The results show that students with MLD showed traces of algebraic thinking, despite their severe difficulties in mathematics. This is particularly interesting taking in consideration that our sample is also composed by a student who attends a special class, where students rarely encounter certain advanced mathematical topics as algebra.

With students with severe difficulties, pushing towards simplification and meaninglessness in favour of technique is not indispensable, nor is it always fruitful. The results of this research show that proposing problems that make use of *algebraic thinking* is possible.

The study presented here is part of an ongoing research project which macro-objective is to understand if and how students with MLD manifest algebraic thinking. We will carry out further analyses in order to describe the algebraic thinking of these students in more detail.

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